Oppg. 19 ***Sylinderkoordinater.*** Et vektorfelt er gitt i sylinderkoordinater

$$\mathbf{v} = u(r,\theta,z)\mathbf{i}_r + v(r,\theta,z)\mathbf{i}_\theta + w(r,\theta,z)\mathbf{k}.$$

Det er også gitt at gradientoperatoren kan skrives

$$\nabla = \mathbf{i}_r \frac{\partial}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{k}\frac{\partial}{\partial z},$$

I denne oppgaven skal vi finne en rekke viktige relasjoner i kurset vha. dyaderegning.

a) Vis at

$$\frac{\partial \mathbf{i}_r}{\partial \theta} = \mathbf{i}_\theta, \quad \frac{\partial \mathbf{i}_\theta}{\partial \theta} = -\mathbf{i}_r.$$

b) Vis

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}.$$

og

$$\nabla^2 \beta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \beta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \beta}{\partial \theta^2} + \frac{\partial^2 \beta}{\partial z^2},$$

der β er et skalarfelt.

c) Vis

$$\begin{aligned}
\nabla \mathbf{v} = \quad & \tfrac{\partial u}{\partial r}\mathbf{i}_r\mathbf{i}_r & + \tfrac{\partial v}{\partial r}\mathbf{i}_r\mathbf{i}_\theta & \quad + \tfrac{\partial w}{\partial r}\mathbf{i}_r\mathbf{k} \\
& + \tfrac{1}{r}\left(\tfrac{\partial u}{\partial \theta} - v\right)\mathbf{i}_\theta\mathbf{i}_r & + \tfrac{1}{r}\left(u + \tfrac{\partial v}{\partial \theta}\right)\mathbf{i}_\theta\mathbf{i}_\theta & \quad + \tfrac{1}{r}\tfrac{\partial w}{\partial \theta}\mathbf{i}_\theta\mathbf{k} \\
& + \tfrac{\partial u}{\partial z}\mathbf{k}\mathbf{i}_r & + \tfrac{\partial v}{\partial z}\mathbf{k}\mathbf{i}_\theta & \quad + \tfrac{\partial w}{\partial z}\mathbf{k}\mathbf{k}
\end{aligned}$$

d) Vis

$$\nabla^2 \mathbf{v} = \left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)\mathbf{i}_r + \left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right)\mathbf{i}_\theta + \left(\nabla^2 w\right)\mathbf{k}.$$

Det kan lønne seg å vise denne for $\mathbf{v} = u\mathbf{i}_r$ og $\mathbf{v} = v\mathbf{i}_\theta$ separat.

e) Vis at tøyningsratetensoren kan uttrykkes

$$\begin{aligned}
\dot{\mathcal{D}} = \quad & \dot{\epsilon}_{rr}\mathbf{i}_r\mathbf{i}_r & + \dot{\epsilon}_{r\theta}\mathbf{i}_r\mathbf{i}_\theta & \quad + \dot{\epsilon}_{rz}\mathbf{i}_r\mathbf{k} \\
& + \dot{\epsilon}_{r\theta}\mathbf{i}_\theta\mathbf{i}_r & + \dot{\epsilon}_{\theta\theta}\mathbf{i}_\theta\mathbf{i}_\theta & \quad + \dot{\epsilon}_{\theta z}\mathbf{i}_\theta\mathbf{k} \\
& + \dot{\epsilon}_{rz}\mathbf{k}\mathbf{i}_r & + \dot{\epsilon}_{\theta z}\mathbf{k}\mathbf{i}_\theta & \quad + \dot{\epsilon}_{zz}\mathbf{k}\mathbf{k}
\end{aligned}$$

der

$$\dot{\epsilon}_{rr} = \tfrac{\partial u}{\partial r}, \qquad \dot{\epsilon}_{r\theta} = \tfrac{1}{2}\left(\tfrac{\partial v}{\partial r} + \tfrac{1}{r}\tfrac{\partial u}{\partial \theta} - \tfrac{v}{r}\right), \quad \dot{\epsilon}_{rz} = \tfrac{1}{2}\left(\tfrac{\partial w}{\partial r} + \tfrac{\partial u}{\partial z}\right),$$

$$\dot{\epsilon}_{\theta\theta} = \tfrac{1}{r}\left(\tfrac{\partial v}{\partial \theta} + u\right), \quad \dot{\epsilon}_{\theta z} = \tfrac{1}{2}\left(\tfrac{1}{r}\tfrac{\partial w}{\partial \theta} + \tfrac{\partial v}{\partial z}\right), \qquad \dot{\epsilon}_{zz} = \tfrac{\partial w}{\partial z}.$$

f) For en inkompressibel væske er spenningstensoren gitt ved

$$\mathcal{P} = -pI + 2\mu\dot{\mathcal{D}}.$$

Finn $\mathbf{P}_\mathbf{n}$ for $\mathbf{n} = \mathbf{i}_r,\ \mathbf{i}_\theta,\ \mathbf{k}$.

g) Vi skriver $\mathcal{P} = p_{rr}\mathbf{i}_r\mathbf{i}_r + ..$ og søker prinsipalretninger og prinsipalspenninger i polarkoordinater. Sett opp egenverdiproblemet som gir disse.

h) Forklar hvorfor dissipasjonen kan uttrykkes

$$\Delta = 2\mu\dot{\mathcal{D}}\cdot\cdot\dot{\mathcal{D}},$$

som gir

$$\Delta = 2\mu\left(\dot{\epsilon}_{rr}^2 + \dot{\epsilon}_{\theta\theta}^2 + \dot{\epsilon}_{zz}^2 + 2(\dot{\epsilon}_{r\theta}^2 + \dot{\epsilon}_{rz}^2 + \dot{\epsilon}_{\theta z}^2),\right)$$