5.9 Find the stiffness and compliance matrices for a UD composite with

$$E_L = 20\,GPa,\quad E_T = 2\,GPa,\quad G_{LT} = 0.7\,GPa,\quad \nu_{LT} = 0.35$$

The compliance matrix $\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}$ is given by:

$$S_{11} = \frac{1}{E_L} = 0.05 \qquad S_{12} = -\frac{\nu_{LT}}{E_L} = -0.0175$$

$$S_{22} = \frac{1}{E_T} = 0.5 \qquad S_{66} = \frac{1}{G_{LT}} = 1.429$$

(all GPa^{-1})

i.e.
$$\begin{bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{bmatrix} = \begin{bmatrix} 0.05 & -0.0175 & 0 \\ -0.0175 & 0.5 & 0 \\ 0 & 0 & 1.429 \end{bmatrix} \begin{bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{bmatrix}$$

The stiffness matrix is the inverse of the compliance matrix:

$$\begin{bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{bmatrix} = \begin{bmatrix} 20.25 & 0.709 & 0 \\ 0.709 & 2.025 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{bmatrix}$$

(all elements in GPa)

5.10 The constraints on elastic constants for orthotropic materials are given in equations (5.82) and (5.85). In (5.85) only the first line applies in 2-D problems.

$$|\nu_{LT}| = 1.97 < \left(\frac{E_L}{E_T}\right)^{1/2} = 3.00 \qquad \text{OK.}$$

$$|\nu_{TL}| = 0.22 < \left(\frac{E_T}{E_L}\right)^{1/2} = 0.33 \qquad \text{OK.}$$

$$E_T,\ E_L > 0 \qquad \text{OK.}$$

An isotropic material is a special case of the orthotropic. If, then, we have an isotropic material with $\nu = 1.97$, we can use the same criteria as for orthotropic materials by setting $\nu_{LT} = \nu_{TL} = \nu$. Equation (5.83) will then be violated:

$$1 - \nu^2 = 1 - 1.97^2 < 0 \quad - \quad \underline{\text{NOT}}\ \text{ok.}$$

5.13
$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = [T]^{-1}\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & 2Q_{66}\end{bmatrix}[T]\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \frac{1}{2}\gamma_{xy}\end{bmatrix}$$

$$= \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23}\\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{66}\end{bmatrix}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}$$

$$[T] = \begin{bmatrix}c^2 & s^2 & 2sc\\ s^2 & c^2 & -2sc\\ -sc & sc & c^2-s^2\end{bmatrix} \qquad \begin{matrix}c = \cos\theta\\ s = \sin\theta\end{matrix}$$

Note that Q_{66} must be multiplied by 2 before the transformation is performed and that $\bar{Q}_{13}$, $\bar{Q}_{23}$ and $\bar{Q}_{66}$ are each one-half of the terms that come from the transformation. This is because we are using the engineering shear strain γ_{xy} whereas the transformation rule applies to the tensor shear strain $\gamma_{xy}/2$

The results are as follows:

$\theta = 30°$:
$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}12.1625 & 4.0375 & 5.8240\\ 4.0375 & 3.1625 & 1.9702\\ 5.8240 & 1.9702 & 4.0375\end{bmatrix}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}$$

$\theta = 45°$:
$$\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix} = \begin{bmatrix}6.55 & 5.15 & 4.5\\ 5.15 & 6.55 & 4.5\\ 4.5 & 4.5 & 5.15\end{bmatrix}\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}$$

$\theta = 60°$:
$$\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix} = \begin{bmatrix}3.1625 & 4.0375 & 1.9702\\ 4.0375 & 12.1625 & 5.8240\\ 1.9702 & 5.8240 & 4.0375\end{bmatrix}\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}$$

5.1 We have $\sigma_x = \tau_{xy} = 0$, $\sigma_y \neq 0$

$$\begin{bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{bmatrix} = [T]\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}c^2 & s^2 & 2sc\\ s^2 & c^2 & -2sc\\ -sc & sc & c^2-s^2\end{bmatrix}\begin{bmatrix}0\\ \sigma_y\\ 0\end{bmatrix} = \begin{bmatrix}s^2\\ c^2\\ sc\end{bmatrix}\sigma_y$$

$$\begin{bmatrix}\epsilon_L\\ \epsilon_T\\ \gamma_{LT}\end{bmatrix} = [S]\begin{bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_L} & -\frac{\nu_{LT}}{E_L} & 0\\ -\frac{\nu_{LT}}{E_L} & \frac{1}{E_T} & 0\\ 0 & 0 & \frac{1}{G_{LT}}\end{bmatrix}\begin{bmatrix}s^2\\ c^2\\ sc\end{bmatrix}\sigma_y$$

$$= \begin{bmatrix}\left(\frac{s^2}{E_L} - \frac{c^2\nu_{LT}}{E_L}\right)\\ \left(-\frac{s^2\nu_{LT}}{E_L} + \frac{c^2}{E_T}\right)\\ \frac{sc}{G_{LT}}\end{bmatrix}\sigma_y$$

$$\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \frac{1}{2}\gamma_{xy}\end{bmatrix} = [T]^{-1}\begin{bmatrix}\epsilon_L\\ \epsilon_T\\ \frac{1}{2}\gamma_{LT}\end{bmatrix} = \begin{bmatrix}c^2 & s^2 & -2sc\\ s^2 & c^2 & 2sc\\ sc & -sc & c^2-s^2\end{bmatrix}\begin{bmatrix}\left(\frac{s^2}{E_L} - \frac{c^2\nu_{LT}}{E_L}\right)\\ \left(-\frac{s^2\nu_{LT}}{E_L} + \frac{c^2}{E_T}\right)\\ \frac{sc}{2G_{LT}}\end{bmatrix}\sigma_y$$

But we also have, when $\sigma_x = \tau_{xy} = 0$,

$$\epsilon_x = -\nu_{yx}\,\sigma_y/E_y, \quad \epsilon_y = \sigma_y/E_y \quad \text{and} \quad \gamma_{xy} = -m_y\,\sigma_y/E_L$$

Then we can compare each of these equations with the respective lines in the matrix equation above.

Consider ϵ_y first:

$$\frac{1}{E_y} = s^2\left(\frac{s^2}{E_L} - \frac{c^2\nu_{LT}}{E_L}\right) + c^2\left(-\frac{s^2\nu_{LT}}{E_L} + \frac{c^2}{E_T}\right) + 2sc.\frac{sc}{2G_{LT}}$$

$$= \frac{s^4}{E_L} + \frac{c^4}{E_T} - 2s^2c^2\frac{\nu_{LT}}{E_L} + \frac{s^2c^2}{G_{LT}}$$

Since $2sc = \sin 2\theta$, this gives

$$\frac{1}{E_y} = \frac{s^4}{E_L} + \frac{c^4}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2 2\theta$$

– agrees with (5.26) in the textbook.

Then we consider ϵ_x:

$$-\frac{\nu_{yx}}{E_y} = c^2\left(\frac{s^2}{E_L} - \frac{c^2\nu_{LT}}{E_L}\right) + s^2\left(-\frac{s^2\nu_{LT}}{E_L} + \frac{c^2}{E_T}\right) - \frac{2s^2c^2}{2G_{LT}}$$

$$\Rightarrow \quad \frac{\nu_{yx}}{E_y} = \left(s^4+c^4\right)\frac{\nu_{LT}}{E_L} - c^2s^2\left(\frac{1}{E_L} + \frac{1}{E_T}\right) + \frac{s^2c^2}{G_{LT}}$$

We use $s^4+c^4 = (s^2+c^2)^2 - 2s^2c^2 = 1-2s^2c^2$

and $2sc = \sin 2\theta$, and obtain eventually

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}}\right)\sin^2 2\theta$$

This agrees with (5.28) since $\dfrac{\nu_{LT}}{E_L} = \dfrac{\nu_{TL}}{E_T}$.

Finally we consider γ_{xy}:

$$-\frac{1}{2}\frac{m_y}{E_L} = sc\left(\frac{s^2}{E_L} - \frac{c^2\nu_{LT}}{E_L}\right) - sc\left(-\frac{s^2\nu_{LT}}{E_L} + \frac{c^2}{E_T}\right) + (c^2-s^2)\frac{sc}{2G_{LT}}$$

$$\Rightarrow \quad m_y = -2E_L\left[\frac{s^3c}{E_L} - \frac{sc^3\nu_{LT}}{E_L} + \frac{s^3c\,\nu_{LT}}{E_L} - \frac{sc^3}{E_T} + (c^2-s^2)\frac{sc}{2G_{LT}}\right]$$

$$= \sin 2\theta\left[-s^2 + c^2\nu_{LT} - s^2\nu_{LT} + \frac{c^2E_L}{E_T} - \frac{(c^2-s^2)}{2G_{LT}}\right]$$

We use $c^2 = 1-s^2$:

$$\Rightarrow \quad m_y = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - s^2\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right]$$

This agrees with (5.32).

5.4 Calculate E_x, G_{xy}, ν_{xy}, m_x and m_y for $\theta = 30°$, $45°$ and $60°$ when $E_L = E_T = 15$ GPa, $G_{LT} = 2.5$ GPa and $\nu_{LT} = \nu_{TL} = 0.20$.

Equation	θ:	$30°$	$45°$	$60°$	
(5.25)	E_x:	8.9552	7.8947	8.9552	(GPa)
(5.36)	G_{xy}:	4.5454	6.25	4.5454	(GPa)
(5.27)	ν_{xy}:	0.5224	0.5789	0.5224	
(5.30)	m_x:	2.9067	2.4563	1.3478	
(5.32)	m_y:	1.3478	2.4563	2.9067	