Chapter: 10

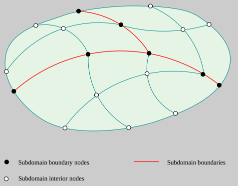
MEK4560
The Finite Element Method in Solid Mechanics II

(April 2, 2008)

Torgeir Rusten

(E-post:torgeiru@math.uio.no)

Department of Mathematics
University of Oslo



Contents

Page 1 of 27

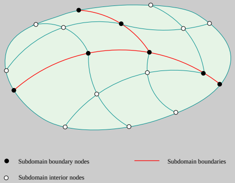
Go Back

Close

Quit

Contents

Department of Mathematics
University of Oslo



Mek 4560
Torgeir Rusten

Contents

Page 2 of 27

Go Back

Close

Quit

10. Linearized Buckling Analysis

An important aspect of the analysis of constructions is to analyze the stability of an equilibrium position. Loosely speaking a stable equilibrium position can be defied as follows:

An equilibrium position of a construction is stable if the construction return to the equilibrium position after a “small” deformation.

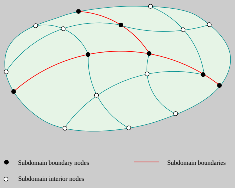
In the present chapter linearized buckling is introduced. We do this on a simple beam problem. The equilibrium equations for a deformed configuration is derived. In Chapter 11 we consider non-linear elasticity in general. The analysis can be based on:

- the equilibrium equation (*the differential equation*),
- the principle of minimum potential energy.

Here we first introduce the equilibrium equation, then the principle of minimum potential energy.

Buckling is loss of stability of an equilibrium configuration, without fracture or degeneration of the material. Often membrane strain energy is converted to bending strain energy with no change in externally applied load.

Department of Mathematics
University of Oslo

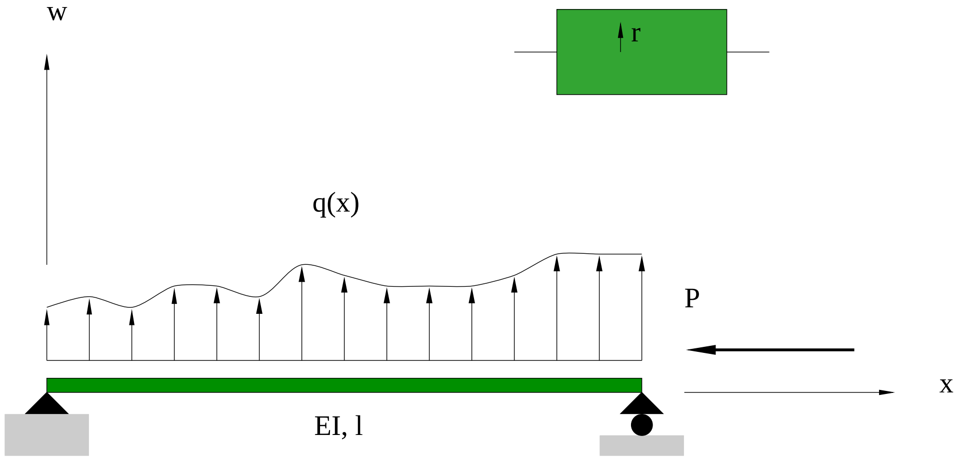


Mek 4560
Torgeir Rusten

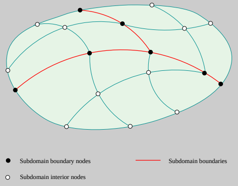
Buckling is discussed in Chapter 18 in [Cook et al., 2002][1].

10.1. Equilibrium in deformed geometry

In order to introduce non-linear models we consider a beam example. The assumption that the difference in configuration before and after deformation is sufficiently large to be taken into account in the model change the equation modeling the beam. Consequently the virtual work is modified and the matrices in the element method is changed.

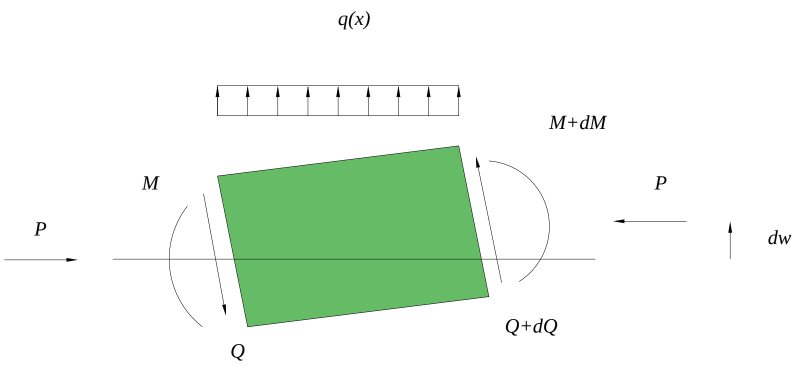


[1] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition, October 2002.



The figure show a beam with a transverse load $q(x)$. The axial force is assumed to be constant, P. The load P is positive for pressure. The transverse displacement is denoted $w = w(x)$, relative to the undeformed beam. The coordinate system is chosen such that axial deformations and bending is decoupled. The axial force P is known. We are interested in the bending part of the equation.

The differential equation for the transverse displacement, $w = w(x)$, are derived in the usual way by considering the equilibrium of an arbitrary beam element shown in the Figure below.



Department of Mathematics
University of Oslo

Mek 4560
Torgeir Rusten

The following assumptions are used in the derivation:

- the sign of the cross sectional forces is indicated on the figure above,
- the rate of change of momentum and transverse force over the element are denoted dM and dQ,
- the axial force is constant, P,
- the transverse load is constant over the element,
- the transverse displacements are sufficiently small for the use of the following approximations:

$$\theta = \sin(\theta) = \tan(\theta) = \frac{dw}{dx}$$

$$\cos(\theta) = 1 \qquad \text{and} \qquad ds = dx$$

The equilibrium in the z-direction takes the form

$$q(x)dx - Q + (Q + dQ) = 0$$

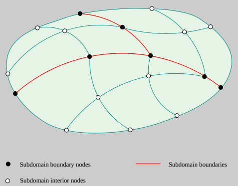
thus the differential equation becomes

$$\frac{dQ}{dx} = -q(x) \tag{10.1}$$

In order to model the equilibrium of the moment the change in geometry is taken into account

$$M - Pdw - Qdx + \frac{1}{2}q(x)dx^2 - (M + dM) = 0$$

Department of Mathematics
University of Oslo



Mek 4560
Torgeir Rusten

resulting in the differential equation

$$\frac{dM}{dx} + P\frac{dw}{dx} = -Q \tag{10.2}$$

Note that the term dx^2 approach zero faster than the other terms when $dx \to 0$, thus the term is neglected. The equation is differentiated and $Q_{,x}$ is substituted from Equation Equation 10.1:

$$\frac{d^2M}{dx^2} + P\frac{d^2w}{dx^2} = q(x) \tag{10.3}$$

We also know that the moment can be expressed using bending, $w_{,xx}$:

$$M = EI\frac{d^2w}{dx^2}$$

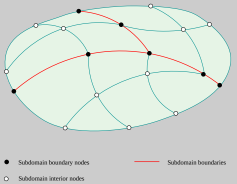
Substitution into Equation Equation 10.3 result in the differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + P\frac{d^2w}{dx^2} = q(x) \tag{10.4}$$

This is the equilibrium equation for a beam loaded by a transverse force and a constant axial force, expressed in the deformed configuration.

If the axial force is zero the equation is simplified to the beam equation derived in *MEK4550, The Finite Element Method in Solid Mechanics I.*

Department of Mathematics
University of Oslo



Mek 4560
Torgeir Rusten

10.2. Virtual work and the Finite Element method

Multiplying the equation (10.4) by an arbitrary displacement δw we obtain

$$\int_0^\ell \Big(\frac{d^2}{dx^2} \Big(EI \frac{d^2 w}{dx^2} \Big) + P \frac{d^2 w}{dx^2} - q(x) \Big) \delta w \, dx = 0$$

where we assume that the beam is parameterized from 0 to the length ℓ. This is the virtual work expression. Using integration by parts it can be reformulated. The virtual work is the stationary value of the potential energy, thus we can write

$$\delta\Pi(w; \delta w) = \int_\ell EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2} - P \int_\ell \frac{dw}{dx} \frac{d\delta w}{dx} + \text{boundary terms} = 0. \qquad (10.5)$$

The first term is well known and model change in stress due to the strain from the stress strain relation, the second term take into account the change in geometry due to an internal stress condition:

- the term

  $$\int_0^\ell EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2}$$

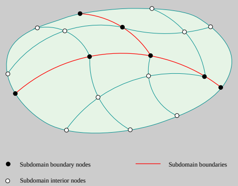
  is the *material stiffness*, and

- the term

  $$P \int_0^\ell \frac{dw}{dx} \frac{d\delta w}{dx}$$

  is the *geometric stiffness*.

Department of Mathematics
University of Oslo



Mek 4560
Torgeir Rusten

Contents
Page 8 of 27
Go Back
Close
Quit

In the Finite Element method the expressions result in the *material stiffness matrix* $\boldsymbol{K}_m$, and the *geometric stiffness matrix*, $\boldsymbol{K}_g$.

Let the transverse load be zero. The stability problem is to concider axial loads resulting in nontrival solutions. Now

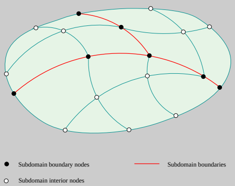
- if the matrix $\boldsymbol{K}_m - P\boldsymbol{K}_g$ has full rank the trivial solution, i.e. $w = 0$ is the only possible solution,
- if $\boldsymbol{K}_m - P\boldsymbol{K}_g$ is singular a non trivial solution exist.

Note that the matrix is singular for the axial forces equal to the eigenvalues of the matrix, i.e. the numbers λ such that

$$\boldsymbol{K}_m = \lambda \boldsymbol{K}_g \tag{10.6}$$

For these numbers an axial force result in a transverse displacement. The is called the linearized buckling problem.

Department of Mathematics
University of Oslo

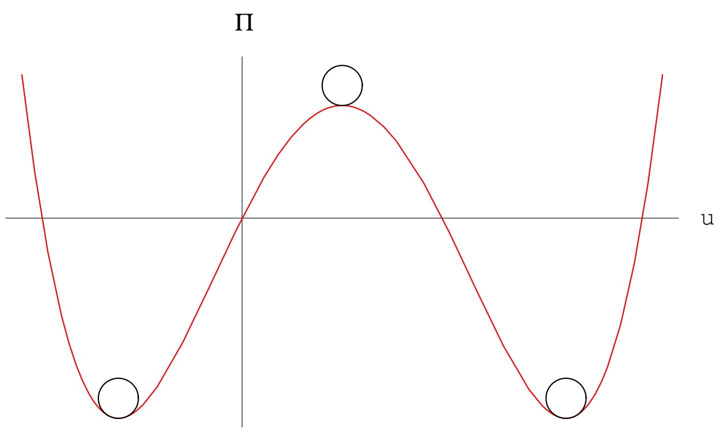


Mek 4560
Torgeir Rusten

10.3. Stability and the potential energy

We have seen that the potential energy functional in linearized elasticity is quadratic, i.e. it is a *convex* functional. Therefore it has only one critical point, and the critical point is a minimum for the functional. Thus, if a construction in equilibrium is perturbed, it will return to the equilibrium position.

For nonlinear problems the potential energy functional may have several critical points, some being local minimum and some might be local maximum. If a construction is in equilibrium at a local maximum, a perturbation it will be in equilibrium at a local minimum. This critical value is unstable, since a small perturbation may result in a large change in the construction. Se also the figure:



10.4. The principle of minimum potential energy

The potential energy functional is the ordinary linear functional plus the axial force times the end displacement:

$$\Pi(w) = \frac{1}{2}\int_0^\ell EI\left(\frac{d^2w}{dx^2}\right)\,dx + Pu(\ell)$$

The axial displacement is a second order effect, due to the transverse displacement of the beam. In linear, firs order, theory this effect is neglected. Here the end displacement must be related to the transverse displacement. When a beam element of length dx is rotated an angle θ, the projection to the beam axis has length $dx\cos\theta$. The local shortening du of the beam becomes

$$du = dx\,(1-\cos\theta) \simeq \frac{1}{2}\theta^2\,dx$$

where we assume that θ is sufficiently small. Introducing:

$$\theta = \frac{dw}{dx} \qquad \text{and} \qquad du = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

The total shortening is found by integration along the beam length:

$$u_\ell = \int_0^\ell du = \int_0^\ell \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

The Potential energy for the beam is:

$$\Pi(w) = \frac{1}{2}\int_0^\ell EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^\ell \left(\frac{dw}{dx}\right)^2 dx$$

10.5. Some comments on the usefulness of linearized buckling

The derivation of the model for transverse displacements of a beam is based on certain assumptions and simplifications. The linearized buckling analysis is based on similar assumptions, thus the analysis is limited to models where the assumptions are valid:

1. Conservative load (derived from a load potential).
2. Deformation up to buckling must be small.
3. Linearly elastic material.
4. The load is independent of the deformation.
5. The construction must be insensitive to imperfections.

10.6. Computations in a linearized buckling analysis

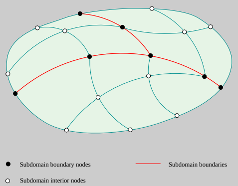
Computation of linearized buckling consist of the following steps:

1. Compute the stiffness matrix, $\boldsymbol{K}_m$, and solve the linear problem

$$\boldsymbol{K}_m \boldsymbol{r} = \boldsymbol{R}(\lambda) = \lambda \boldsymbol{q}$$

where λ is a load level. Find the initial distribution of the stresses.

Department of Mathematics University of Oslo



Mek 4560
Torgeir Rusten

2. Compute a geometric stiffness matrix relative to the computed stresses, $\boldsymbol{K}_g = \lambda \boldsymbol{K}_1$.

3. Solve the eigenvalue problem:

$$(\boldsymbol{K}_m - \lambda \boldsymbol{K}_1)\boldsymbol{r} = \boldsymbol{0}$$

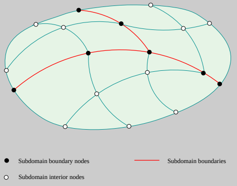
   The eigenvalue with smallest magnitude, i.e. absolute value, λ, is the multiplier giving the buckling load, $P_{critical} = \lambda q$.

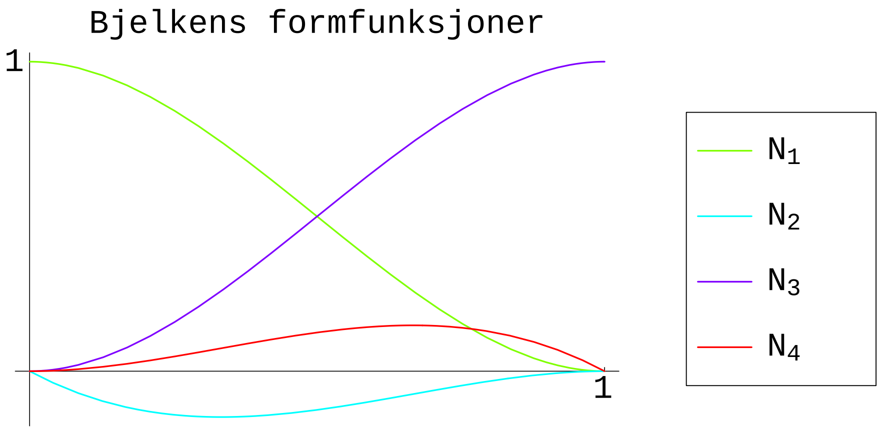
Example. Buckling of a beam: We consider the beam formulation introduced above, and do a buckling analysis for a simply supported beam.

We use a Finite element method. The basis functions are the Hermite functions:

$$w = \boldsymbol{N}\boldsymbol{d} = \left\{ \begin{array}{cccc} \frac{(x-\ell)^2\,(2\,x+\ell)}{\ell^3} & -\left(\frac{x\,(x-\ell)^2}{\ell^2}\right) & \frac{x^2\,(-2\,x+3\,\ell)}{\ell^3} & \frac{x^2\,(-x+\ell)}{\ell^2} \end{array} \right\} \left\{ \begin{array}{c} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{array} \right\}$$

The functions are depicted below.

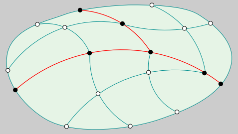




For a beam of length $\ell = 1$ and bending stiffness $EI = 1$ we obtain the material stiffness matrix:

$$\boldsymbol{K}_m = \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix}$$

Department of Mathematics University of Oslo



Mek 4560
Torgeir Rusten

The geometric stiffness matrix becomes:

$$\boldsymbol{K}_g = \frac{1}{30}\begin{bmatrix} 36 & -3 & -36 & -3 \\ -3 & 4 & 3 & -1 \\ -36 & 3 & 36 & 3 \\ -3 & -1 & 3 & 4 \end{bmatrix}$$

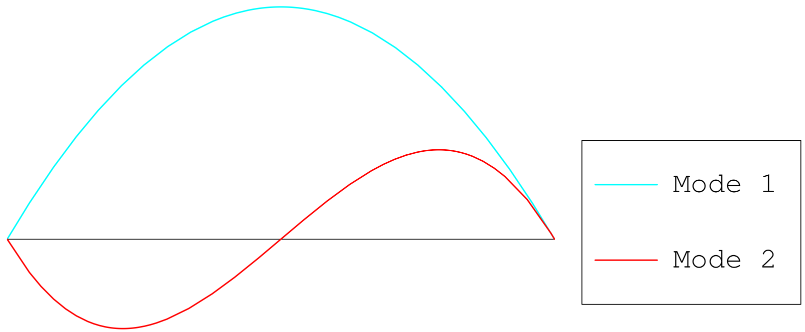
The two end displacements are set to zero. The eigenvalues are:

$$P_{FEM} = \{12 \qquad 60\}$$

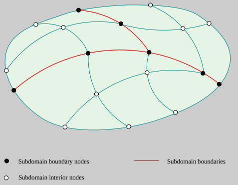
The exact eigenvalues are:

$$P_{eksakt} = \{\pi^2 \qquad 4\pi^2\}$$

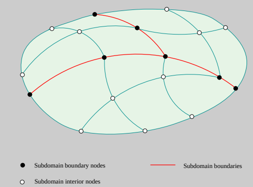
The two buckling form are shown below.



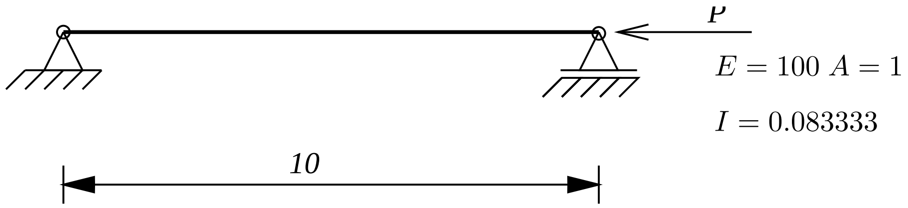
Department of Mathematics University of Oslo



Mek 4560
Torgeir Rusten



10.7. Euler-søyle (BEAM3)



Problem: Foreta en linearisert knekningsanalyse av staven/søylen vist i figuren. Plot de fire første knekkformene. Stemmer knekklastene med de fra den analytiske løsningen?

Løsning:

```
/BATCH, LIST
/FILNAM, EX421
/TITLE, Euler-stav

/PREP7
ET,1,3
R,1,1,0.083333,1
MP,EX,1,100
N                    ! "Direct generation"
N,11,10              ! Knutepunkt på den andre enden
FILL                 ! Ni knutepunkt mellom 1. og 11.
E,1,2                ! 1. element
```

Contents

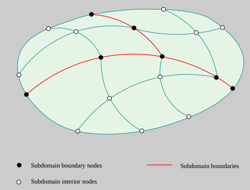
Page 16 of 27

Go Back

Close

Quit

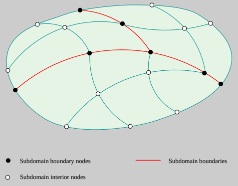
```
EGEN,10,1,1            ! Kopierer 1. elem.
FINI

/SOLU
PSTRES,ON              ! Generering av den geometriskse stivhetsmatrisen
D,1,UX,,,,,UY
D,11,UY
F,11,FX,-1
SOLVE
FINI

/SOLU
ANTYP,BUCK             ! Knekningsanalyse
BUCOPT,SUBS,4
SUBOPT,,10
MXPAND,4               ! 4 knekkformer og egenverdier
SOLVE
FINISH

/POST1
/SHOW,ex421pl          ! Alle plotter til filen ex421pl
!/OUT,form421          ! Utskrift til filen form421
!SET,LIST              ! Gir egenverdiene
!/OUT                  ! Utskrift til default (skjermen) igjen
/WINDOW,1,LTOP         ! Følgende viser de 4 knekkformer i samme vindu
/WINDOW,2,RTOP
/WINDOW,3,LBOT
/WINDOW,4,RBOT
/EDGE,ALL,1
/TRIAD,OFF
/PLOPTS,FRAME,ON
/PLOPTS,INFO,OFF
/PLOPTS,TITLE,OFF
/WINDOW,ALL,OFF
*DO,I,1,4              ! Løkke
  /WINDOW,I,ON
  SET,1,I
```

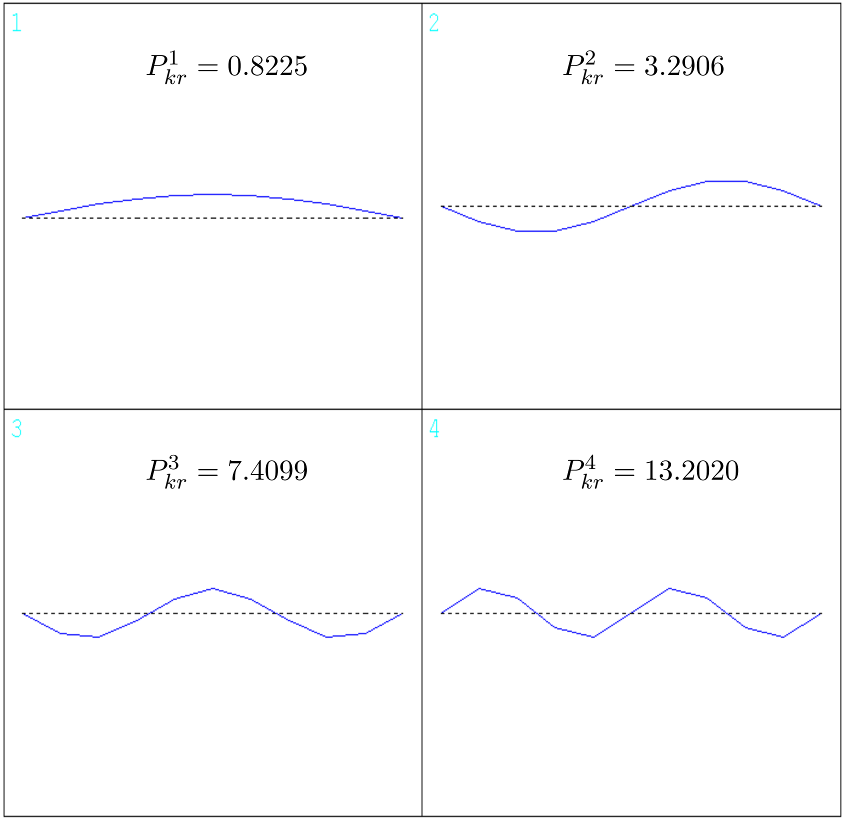



```
  PLDISP,1
  /NOERASE
  /WINDOW,I,OFF
*ENDDO
/WINDOW,ALL,DELE
/RESET
/SHOW,TERM           ! Alle plotter til skjermen igjen
/WINDOW,1,FULL
FINI
EXIT
```

Svar/kommentarer: Som det ble forklart i subsection 10.5 søkes det først $(\boldsymbol{K}_\sigma)_{init}$ ved å påføre lasten $\boldsymbol{R}_{init}$. $\boldsymbol{R}_{init}$ i dette eksemplet er $P = 1$. Når $(\boldsymbol{K}_\sigma)_{init}$ er funnet ved kommandoen `PSTRES, ON` i `/SOLU` som innmatningsfilen ovenfor viser.

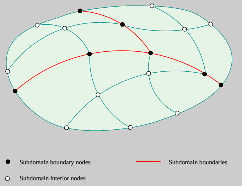
Det er brukt **APDL** (ANSYS Parametric Design Language) løkke (`*DO`) for fremstilling av alle 4 knekkformer i et skjermvindu.





Knekkformer for en Euler-søyle

Department of Mathematics University of Oslo

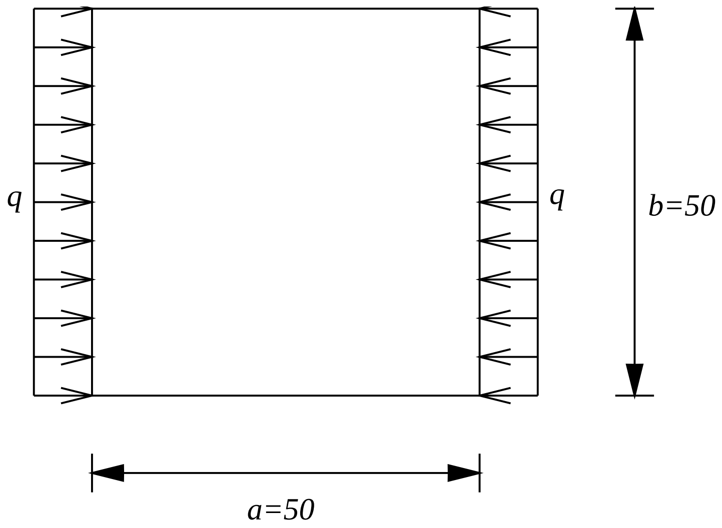


Mek 4560
Torgeir Rusten

$\nu = 0.3$

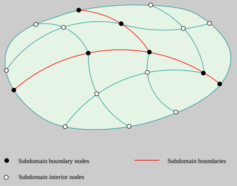
$h = 0.5$

10.8. Fritt opplagt plate (SHELL63)



Problem: Platen som er vist i figuren er fritt opplagt på alle kanter. Den er belastet med en jevnt fordelt trykk q på to motsatte kanter som figuren viser. Materialegenskapene er gitt

Department of Mathematics University of Oslo



Mek 4560
Torgeir Rusten

ved: $E = 30 \times 10^6$ og $\nu = 0.3$.

[Crisfield, 1991] gir at q_{kr} (kritisklast pr. lengdeenhet) er gitt ved

$$q_{kr} = \frac{k\pi^2 E h^3}{12(1-\nu^2)b^2}$$

hvor $k = 4.0$, h er platetykkelsen og dermed $q_{kr} = 5422.86$.

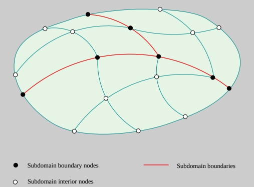
Foreta en elementmetodeanalyse og undersøk om q_{kr} stemmer med verdien ovenfor. Hva kan være årsak til en eventuell forskjell i q_{kr}.

Plott de fire første knekkformer i samme figur.

Løsning:

```
/BATCH, LIST
/FILNAM, ex422
/TITLE, Fritt opplagt plate
/VIEW,,1,1,1
/ANGLE,,-90,XM,1
/ANGLE,,90,ZM,1

/PREP7
ET,1,63
R,1,0.5              ! Platetykkelsen
MP,EX,1,30e6
MP,NUXY,1,0.3        ! Poissons tall
K,1
K,2,,50
```

```
K,3,50,50
K,4,50
A,1,2,3,4
KESIZE,all,5
AMESH,1
EPLO
FINISH
/SOLU
ANTYPE,STAT
PSTRES,ON
/PBC,ALL,1            ! Vis grensebetingelsene osv.
NSEL,S,LOC,X
NSEL,A,LOC,Y
NSEL,A,LOC,Y,50
D,ALL,UZ
NSEL,S,LOC,X,50
D,ALL,UX,,,,,UZ
NSEL,ALL
D,NODE(0,25,0),UY $ D,NODE(50,25,0),UY
/PSF,PRES,NORM,2
NSEL,S,LOC,X
ESLN,S,0
SFE,ALL,3,PRES,,1
ALLSEL
EPLO
SOLVE
FINISH
/SOLU
ANTYPE,BUCK,NEW
BUCOPT,SUBS,5
SUBOPT,,4
MXPAND,4
SOLVE
FINISH

/POST1
/SHOW,ex422pl         ! Alle plotter til filen ex421pl
```

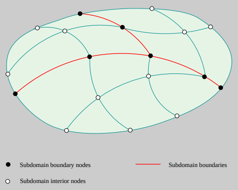



```
!/OUT,form421         ! Utskrift til filen form421
!SET,LIST             ! Gir egenverdiene
!/OUT                 ! Utskrift til default (skjermen) igjen
FINI
```

Svar/kommentarer ANSYS kjøringen gir $P^1_{kr} = 5391.8$ som stemmer med den analytiske løsningen. Se forøvrig [Crisfield, 1991] for de antagelser som er gjort ved utledningen av det analytiske uttrykket for P^1_{kr}. Den litt lavere verdien som ANSYS gir for P^1_{kr} i forhold til den analytiske kan forklares med at det er brukt redusert integrasjon [Cook et al., 1989, Bathe, 1982] i beregningene.

Legg også merke til fastsettelsen av grensebetingelsene - det må ikke forekomme fritt legeme bevegelser.

Figuren nedenfor viser de 4 første knekklaster og knekkformer for platen.



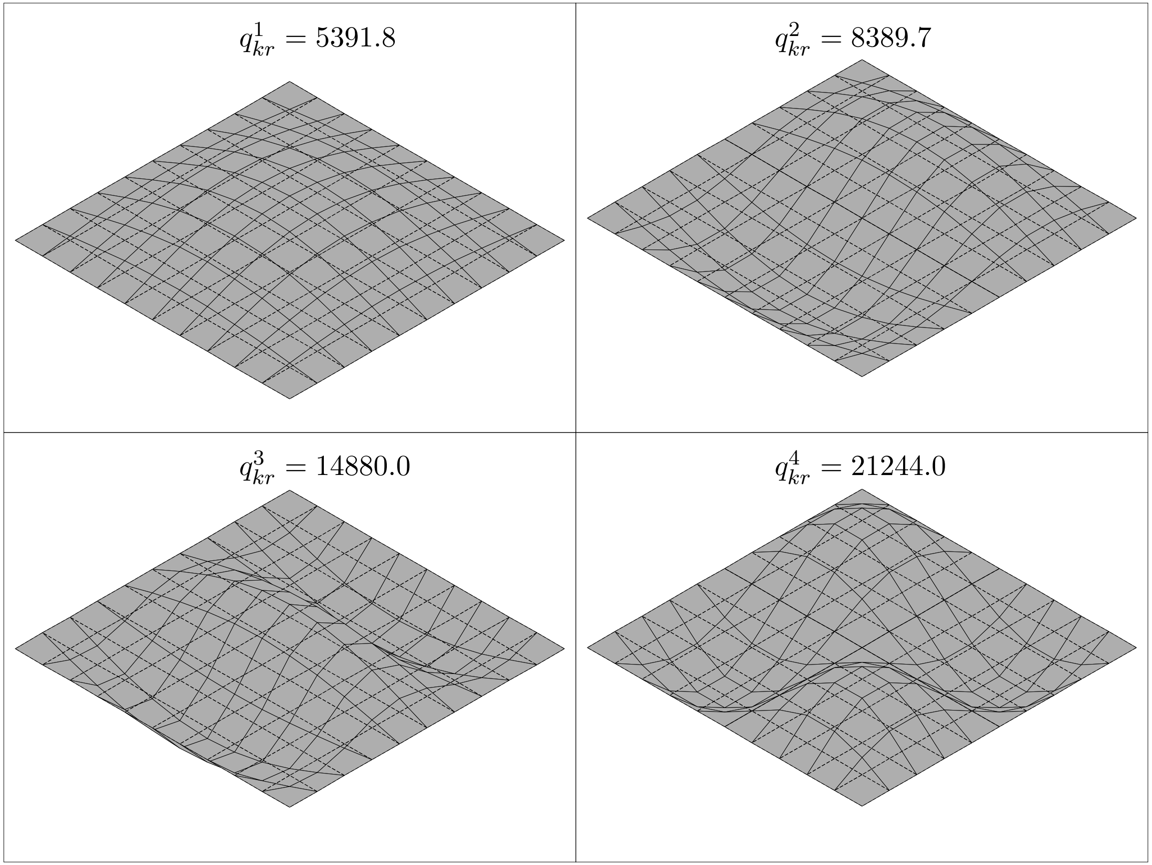
Contents

Page 23 of 27

Go Back

Close

Quit



Knekkformer for en fritt opplagt plate

Department of Mathematics
University of Oslo

Mek 4560
Torgeir Rusten

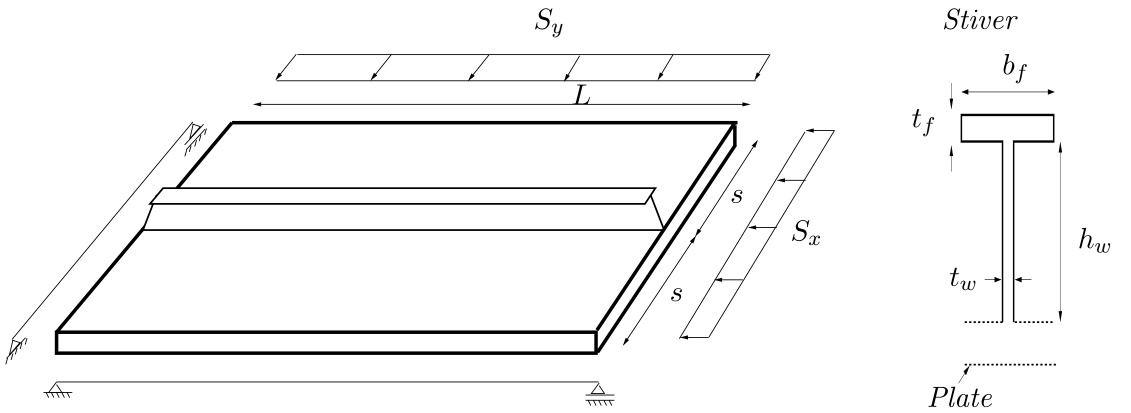


Figure 10.1: Fritt opplagt plate med en stiver plassert på midten.

Øving 10.1

Platen har tykkelse $t = 18\,mm$, lengde $L = 2400\,mm$, og en avstand fra randen til stiveren lik $s = 800\,mm$. Platens er fritt opplagt, og rendene antas å forbli rette under deformasjonen p.g.a en omkringliggende konstruksjon. Dimensjonene til stiveren: $t_f = 18\,mm$, $b_f = 100\,mm$, $t_w = 10\,mm$ og $h_w = 300\,mm$. S_x og S_x er den ytre gjennomsnittspenningen påsatt platens rand. Stiveren er ikke belastet med en ytre spenning.

a) Foreta en serie av egenverdiberegninger (gjerne v.h.a. en DO-løkke), og tegn opp interaksjonskurver for den elastiske knekklasten i $S_x - S_y$ planet, f.eks

$$(S_x, S_y) = \lambda \{(1,0), \cdots, (1,1), \cdots, (0,1), \cdots, (-1,1), \cdots, (-1,0), \cdots, (-1,1), \cdots\}$$

b) Lag et plot av knekkformen når den kun er belastet i x-retning (dvs. $S_y = 0$).

A. References

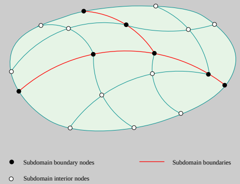
[Bathe, 1982] Bathe, K. J. (1982). *The Finite Element Procedures in Engineering Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey.

[Cook et al., 1989] Cook, R. D., Malkus, D. S., and Plesha, M. E. (1989). *Concepts and Applications of Finite Element Analysis*. John Wiley & Sons, Inc., 3rd edition.

[Cook et al., 2002] Cook, R. D., Malkus, D. S., Plesha, M. E., and Witt, R. J. (2002). *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition.

[Crisfield, 1991] Crisfield, M. (1991). *Non-Linear Finite Element Analysis of Solids and Structures, Volume 1: Essentials*. Wiley.

Department of Mathematics University of Oslo



Mek 4560
Torgeir Rusten