# *Chapter: 2*

# MEK4560
# The Finite Element Method in Solid Mechanics II

(April 17, 2008)


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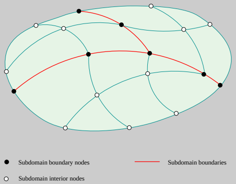

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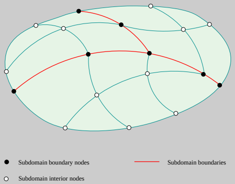


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# 2. *Mindlin-Reissner* beam

The purpose of this chapter is to derive the *Mindlin-Reissner* beam model. It includes transverse shear deformations and are important for short/wide beams. (Why?)

The model is derived in three dimensions and based on the general formulation of three dimensional elasticity. The derivation of some plate models are similar. The models are mentioned in [Bell, 1994][1] and [Cook et al., 2002][2], but the derivation is not discussed in much detail. A derivation similar to the following is found in [Hughes, 1987][4]

[1] Kolbein Bell. *Matrisestatikk*. Number ISBN: 82-519-1162-1 (ib.). Tapir, 1994.

[2] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition, October 2002.

[4] T. J. R. Hughes. *The Finite Element Method, Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

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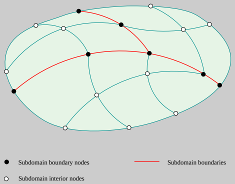


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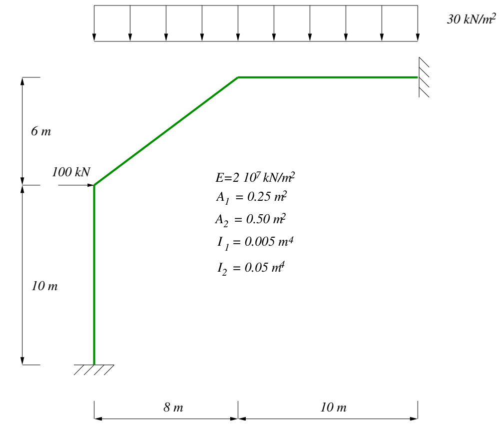
## 2.1. Assumptions

**Geometry:** The construction is a set of “straight line segments” connected in nodes at the endpoints of the line segments. Each line segment has a local “x” axis along the line and local “y” and “z” axis normal to the “x” axis. An example is shown in the figure below.




We assume that the distance between the nodes, i.e. the length of the beams along the “x” axis, are “large” compared to the cross sectional dimension of the beam along the “y” and “z”.

$$V = \bigcup_{e=1}^{n_e} V^e$$
$$V^e = \left\{(x, y, z) \in \mathbb{R}^3 | x \in [0, \ell^e], (y, z) \in A^e \subset \mathbb{R}^2\right\}$$

where $\ell$ is the length and $A$ the cross section of the beam. We assume that the beam is prismatic, but it is trivial to take into account that $A^e = A^e(x)$.

**Stresses:** We assume that the normal and tangential stress on planes parallel to the beam axis, i.e.

$$\sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0.$$

These assumptions is used to eliminate $\varepsilon_{yy}$, $\varepsilon_{zz}$ and $\varepsilon_{yz}$. To be precise, the first two are expressed in terms of $\varepsilon_{xx}$

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx} \tag{2.1}$$

while, $\varepsilon_{yz} = 0$. (Show this!)

**Displacements:** We assume that plane cross sections initially normal to the beam axis remains plane after deformation but not necessarily normal to the deformed axis. This can be written

$$\begin{aligned} u(x,y,z) &= u_0(x) + z\theta_y(x) - y\theta_z(x) \\ v(x,y,z) &= v_0(x) - z\theta_x(x) \\ w(x,y,z) &= w_0(x) + y\theta_x(x) \end{aligned} \tag{2.2}$$

The rotation, $\theta_i$, are defined using the right hand rule. The functions to be found in the beam formulation are

$$\boldsymbol{u}^T = (u_0, v_0, w_0, \theta_x, \theta_y, \theta_z)$$

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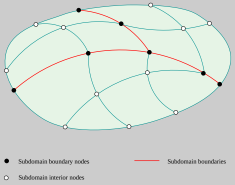


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**Stress-strain relations:** For an homogeneous, isotropic material the stress strain relations on tensor form are

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

where $\lambda$ and $\mu$ is Lamès konstanter. The assumptions on zero stress was used above to express $\varepsilon_{yy}$ and $\varepsilon_{zz}$ in terms of $\varepsilon_{yz}$. Using this it can be shown that

$$\sigma_{xx} = E\varepsilon_{xx}$$
$$\sigma_{xy} = 2G\varepsilon_{xy} \quad \text{and} \quad \sigma_{xz} = 2G\varepsilon_{xz}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = G = \frac{E}{2(1+\nu)}$$

In the notation used in Chapter 1 we have

$$\tau_{xy} = G\gamma_{xy} \quad \text{and} \quad \tau_{xz} = G\gamma_{xz} \tag{2.3}$$

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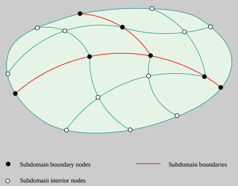


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**Strains:** Substitution of the displacement relations Equation 2.2 the strain relations result in

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = u_0' + z\theta_y' - y\theta_z' \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = 0 \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\theta_z + v_0' - z\theta_x' \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\theta_x + \theta_x = 0 \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta_y + w_0' + y\theta_x'
\end{aligned}
$$

where the notation $'$ is used for $\frac{d}{dx}$.

**Remark 2.1** Note that there is an inconsistency in the relation between stress and strains. The material law states that

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$$

while the assumptions on the displacements results in the strains $\varepsilon_{yy} = \varepsilon_{zz} = 0$.

## 2.2. Potential energy

The usual potential energy function is:

$$\Pi(\boldsymbol{u}) = \frac{1}{2}\int_V \sigma^T \varepsilon(\mathbf{u})\, dV - \int_V \boldsymbol{u}^T \boldsymbol{F}\, dV - \int_{S_t} \boldsymbol{u}^T \boldsymbol{\Phi}\, dS$$

In the follwing we neglect the last term, the treatment is straightforward if the term is required.

We now consider one beam element and assume that the $x$ axis coincide with the beam axis and that the cross section is aligned with the $y$ and $z$ axis. The the integral over the volume becomes

$$\int_{dV^e} \{\}\, dV = \int_0^{\ell^e} \int_{A^e} \{\}\, dA\, dx \tag{2.4}$$

Inserting the displacements and strains we obtain, where we neglect the superscripts $e$:

$$\begin{aligned}
&\Pi(u_0, v_0, w_0, \theta_x, \theta_y, \theta_z) = \\
&\quad \frac{1}{2}\int_0^{\ell}\int_A \sigma_{xx}\left(u_0' + z\theta_y' - y\theta_z'\right) + \sigma_{xy}\left(v_0' - z\theta_x' - \theta_z\right) + \sigma_{xz}\left(w_0' + y\theta_x' + \theta_y\right)\, dA\, dx \\
&\qquad - \int_0^{\ell}\int_A F_x\left(u_0 + z\theta_y - y\theta_z\right) + F_y\left(v_0 - z\theta_x\right) + F_z\left(w_0 + y\theta_x\right)\, dA\, dx
\end{aligned}$$

In order to proceed the following quantities are introduced

$$
\begin{aligned}
N &= \int_A \sigma_{xx} \, dA && \textit{Axial force} \\
M_y &= \int_A \sigma_{xx} z \, dA && \textit{Bending moment} \\
M_z &= \int_A \sigma_{xx} y \, dA && \textit{Bending moment} \\
T &= \int_A \sigma_{xz} y - \sigma_{xy} z \, dA && \textit{Twisting moment} \\
Q_y &= \int_A \sigma_{xy} \, dA && \textit{Shear forces} \\
Q_z &= \int_A \sigma_{xz} \, dA && \textit{Shear forces}
\end{aligned}
$$

and the strain energy becomes

$$
\frac{1}{2} \int_0^\ell N u_0' + Q_y \left( v_0' - \theta_z \right) + Q_z \left( w_0' + \theta_y \right) + M_y \theta_y' - M_z \theta_z' + T \theta_x' \, dx
$$

where

$$
\begin{aligned}
\varepsilon_0 &= u_0' && \textit{Axial strain} \\
\gamma_y &= v_0' - \theta_z && \textit{Shear strain} \\
\gamma_z &= w_0' + \theta_y && \textit{Shear strain} \\
\kappa_y &= \theta_z' && \textit{Bending} \\
\kappa_z &= \theta_y' && \textit{Bending} \\
\Psi &= \theta_x' && \textit{Torsion}
\end{aligned}
$$

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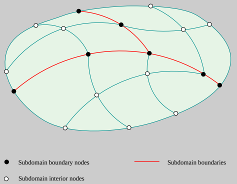


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**Stress-strain relations for beams:** Using the above we express the stress-strain relations using quantities relevant for the beam model.

## Axial force:

$$N = \int_A \sigma_{xx}\, dA = \int_A E\left(u_0' + z\theta_y' - y\theta_z'\right)\, dA = EAu_0' + ES_z\theta_y' - ES_y\theta_z'$$

where $S_y = \int_A y\, dA$ and $S_z = \int_A z\, dA$. Note that if the beam axis is in the centroid of the cross section[1], we obtain the usual axial force

$$N = EAu_0'$$

**Bending moments:** Bending with respect to the $y$-axis:

$$M_y = \int_A \sigma_{xx} z\, dA = \int_A Ez\left(u_0' + z\theta_y' - y\theta_z'\right)\, dA = ES_z u_0' + EI_{zz}\theta_y' - EI_{yz}\theta_z'$$

Similarly, the bending moment with respect to the $z$-axis

$$M_z = \int_A \sigma_{xx} y\, dA = \int_A Ey\left(u_0' + z\theta_y' - y\theta_z'\right)\, dA = ES_y u_0' + EI_{yz}\theta_y' - EI_{yy}\theta_z'$$

[1] If the beam axis is in the centroid of the cross section: (Why?)

$$\int_A y\, dA = \int_A z\, dA = 0.$$

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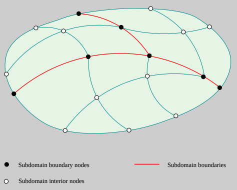


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where the second moments of inertia are $I_{yy} = \int_A y^2\,dA$, $I_{zz} = \int_A z^2\,dA$ and $I_{yz} = \int_A yz\,dA$.

The torsional moment can also be expressed using the kinematics ant the material law:

$$\begin{aligned} T &= \int_A (\sigma_{xz} y - \sigma_{xy} z)\ dA = \int_A \left( Gy(\theta_y + w_0' + y\theta_x') + Gz(\theta_z - v_0' + z\theta_x') \right)\ dA \\ &= GS_y(\theta_y + w_0') + GS_z(\theta_z - v_0') + GI_{xx}\theta_x' \end{aligned}$$

where

$$I_{xx} = I_{yy} + I_{zz} = \int_A (y^2 + z^2)\,dA$$

If the beam axis is at the centroid of $A$ we obtain

$$T = GI_{xx}\theta_x', \qquad M_y = E(I_{zz}\theta_y' - I_{zy}\theta_z') \qquad \text{and} \qquad M_z = E(I_{yz}\theta_y' - I_{yy}\theta_z')$$

**Shear forces:** The shear forces are given by

$$Q_y = \int_A \sigma_{xy}\,dA = \int_A G(v_0' - \theta_z - z\theta_x')\,dA = GA\gamma_y - GS_z\theta_x'$$

and

$$Q_z = \int_A \sigma_{xz}\,dA = \int_A G(w_0' + \theta_y + y\theta_x')\,dA = GA\gamma_z + GS_y\theta_x'$$

Again, by proper placement of the $x$-axis

$$Q_y = GA\gamma_y = GA_y^s\gamma_y \qquad \text{and} \qquad Q_z = GA\gamma_z = GA_z^s\gamma_z$$

where $A_i^s$ is the called the effective shear area.

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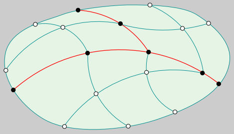




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**Strain energy:** Using this the strain energy takes the form

$$\int_0^\ell (EAu_0' + ES_z\theta_y' - ES_y\theta_z')u_0'\,dx$$
$$+ \int_0^\ell (GA_y^s(v_0' - \theta_z) - GS_z\theta_x')(v_0' - \theta_z)\,dx + \int_0^\ell (GA_z^s(w_0' + \theta_y) + GS_y\theta_x')(w_0' + \theta_y)\,dx$$
$$+ \int_0^\ell (ES_zu_0' + EI_{zz}\theta_y' - EI_{yz}\theta_z')\theta_y'\,dx - \int_0^\ell (ES_yu_0' + EI_{yz}\theta_y' - EI_{yy}\theta_z')\theta_z'\,dx$$
$$+ \int_0^\ell (GS_y(\theta_y + w_0') + GS_z(\theta_z - v_0') + GI_{xx}\theta_x')\theta_x'\,dx$$

Note that the functional is greatly simplified if the the coordinate system is oriented properly:

$$\int_0^\ell (EAu_0'u_0' + GA_y^s(v_0' - \theta_z)(v_0' - \theta_z) + GA_z^s(w_0' + \theta_y)(w_0' + \theta_y)$$
$$+ EI_{zz}\theta_y'\theta_y' - 2EI_{yz}\theta_y'\theta_z' + EI_{yy}\theta_z'\theta_z' + GI_{xx}\theta_x'\theta_x')\,dx$$

This can also be expressed using strains

$$\int_0^\ell EA\varepsilon_0\varepsilon_0 + GA_y^s\gamma_y\gamma_y + GA_z^s\gamma_z\gamma_z + EI_{zz}\kappa_z\kappa_z - 2EI_{yz}\kappa_y\kappa_z + EI_{yy}\kappa_y\kappa_y + GI_{xx}\Psi\Psi\,dx$$

Note that the bending part of the equation can be written

$$\begin{pmatrix}\kappa_y & -\kappa_z\end{pmatrix}\begin{pmatrix}I_{yy} & I_{yz}\\ I_{yz} & I_{zz}\end{pmatrix}\begin{pmatrix}\kappa_y\\ -\kappa_z\end{pmatrix} \tag{2.5}$$

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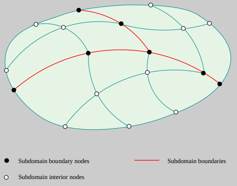


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The matrix can be diagonalized using the eigenvectors. The axis with origin at the $x$-axis and the eigenvectors as directions are the *principal axes*. Using them decouples the bending around the $y$ and $z$ axis.

**Load vector:** We consider volume loads, other loads are similar:

$$\int_0^\ell \int_A F_x \left(u_0 + z\theta_y - y\theta_z\right) + F_y \left(v_0 - z\theta_x\right) + F_z \left(w_0 + y\theta_x\right)\, dAdx$$
$$= \int_0^\ell F_x \left(Au_0 + S_z\theta_y - S_y\theta_z\right) + F_y \left(Av_0 - S_z\theta_x\right) + F_z \left(Aw_0 + S_y\theta_x\right) dx$$

If the beam axis are aligned as above

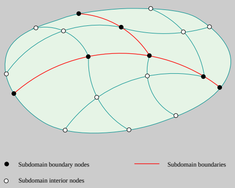
$$W = \int_0^\ell q_x u_0 + q_y v_0 + q_z w_0 \, dx$$

See [Hughes, 1987][4] for further derivation of load vector, see also the notes for *MEK4550, The Finite Element Method in Solid Mechanics I*, Chapter 7.

> *Note that the expressions only involves first derivatives, thus $C^0$ basis functions can be used in a finite element approximation.*

[4] T. J. R. Hughes. *The Finite Element Method, Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

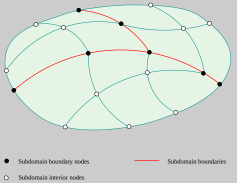




**A square prismatic beam:** Let the cross section be square with length $t$ in each direction. Then $A = t^2$ and $I_{xx} = I_{yy} = t^4/12$. When $t$ becomes small, i.e. the beam is slender, the bendig deformations dominate. However, in the model the the bending terms becomes much smaller than the shear term. Consequently, the shear term will dominate and so called "locking" might be observed in computations.

Note that since the terms in front of $\kappa_z \kappa_z$ and $\kappa_y \kappa_y$ are small, small changes in data, e.g. in loads, may result in a large change in the displacement. This lack of stability is typical for slender beams, plate and shell models, and a careful choice of numerical methods are essential in order to compute accurate results.





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## 2.3. Element stiffness matrices

We assume that all functions in the beam model are interpolated using the same basis functions for all the six degrees of freedom per node:

$$u_0 = \boldsymbol{N}_0 \boldsymbol{d}_u, \qquad v_0 = \boldsymbol{N}_0 \boldsymbol{d}_v, \qquad w_0 = \boldsymbol{N}_0 \boldsymbol{d}_w,$$
$$\theta_x = \boldsymbol{N}_0 \boldsymbol{d}_{\theta_x}, \qquad \theta_y = \boldsymbol{N}_0 \boldsymbol{d}_{\theta_y}, \qquad \theta_z = \boldsymbol{N}_0 \boldsymbol{d}_{\theta_z}.$$

The element displacement vector is

$$\boldsymbol{d} = \begin{Bmatrix} \boldsymbol{d}_u \\ \boldsymbol{d}_v \\ \boldsymbol{d}_w \\ \boldsymbol{d}_{\theta_x} \\ \boldsymbol{d}_{\theta_y} \\ \boldsymbol{d}_{\theta_z} \end{Bmatrix} \quad \text{such that} \quad \boldsymbol{u} = \boldsymbol{N}\boldsymbol{d} \quad \text{and} \quad \boldsymbol{N} = \begin{pmatrix} \boldsymbol{N}_0 & & & & & \\ & \boldsymbol{N}_0 & & & & \\ & & \boldsymbol{N}_0 & & & \\ & & & \boldsymbol{N}_0 & & \\ & & & & \boldsymbol{N}_0 & \\ & & & & & \boldsymbol{N}_0 \end{pmatrix}$$

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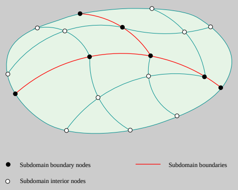

**Strains:** Using the the above derivation of the beam model the the strain operator becomes:

$$\varepsilon_B = \begin{Bmatrix} \varepsilon_0 \\ \gamma_y \\ \gamma_z \\ \kappa_y \\ \kappa_z \\ \Psi \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{\partial}{\partial x} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} = \boldsymbol{\partial u}$$

Inserting the finite element interpolation of $\boldsymbol{u}$ the matrix $\boldsymbol{B}$ becomes:

$$\boldsymbol{\varepsilon} = \boldsymbol{\partial u} = \boldsymbol{\partial N d} \qquad \text{thus} \qquad \boldsymbol{B} = \boldsymbol{\partial N}$$

Inserting the basis functions result in:

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{N}_0' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{N}_0' & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{N}_0 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{N}_0' & \mathbf{0} & \boldsymbol{N}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{N}_0' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{N}_0' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{N}_0' & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

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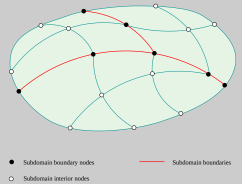


**Material data:** The material data can also be collected in a matrix:

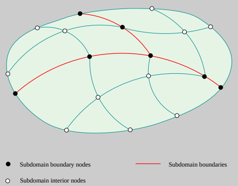
$$\boldsymbol{\sigma}_B = \begin{pmatrix} N \\ Q_y \\ Q_z \\ M_z \\ M_y \\ T \end{pmatrix} = \begin{pmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GA_y^s & 0 & 0 & 0 & 0 \\ 0 & 0 & GA_z^s & 0 & 0 & 0 \\ 0 & 0 & 0 & EI_{yy} & 0 & 0 \\ 0 & 0 & 0 & 0 & EI_{zz} & 0 \\ 0 & 0 & 0 & 0 & 0 & GI_{xx} \end{pmatrix} \begin{pmatrix} \varepsilon_0 \\ \gamma_y \\ \gamma_z \\ \kappa_y \\ \kappa_z \\ \Psi \end{pmatrix} = \boldsymbol{E}_B \boldsymbol{\varepsilon}_B$$

If the beam axis are not placed at the centroid:

$$\boldsymbol{E}_B = \begin{bmatrix} EA & 0 & 0 & -ES_y & ES_z & 0 \\ 0 & GA_y^s & 0 & 0 & 0 & -GS_z \\ 0 & 0 & GA_z^s & 0 & 0 & GS_y \\ ES_y & 0 & 0 & -EI_{yy} & EI_{yz} & 0 \\ ES_z & 0 & 0 & -EI_{yz} & EI_{zz} & 0 \\ 0 & -GS_z & GS_y & 0 & 0 & GI_{xx} \end{bmatrix}$$

Note that the matrix is unsymmetric. This is due to the choice of sign on $M_y$. If the sign is changed the matrix becomes symmetric.





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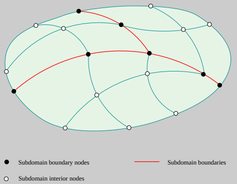


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**The stiffness matrix:** We are now in the position to establish the stiffness matrix as usual:

$$\boldsymbol{k} = \int_0^\ell \boldsymbol{B}^T \boldsymbol{E}_B \boldsymbol{B}\, dx$$

Note that it can be split in four terms

$$\boldsymbol{k} = \underset{\text{axial}}{\boldsymbol{k}_a} + \underset{\text{bending}}{\boldsymbol{k}_b} + \underset{\text{shear}}{\boldsymbol{k}_s} + \underset{\text{torsion}}{\boldsymbol{k}_t}$$

**Remark 2.2** An alternative mehtod used to model shear deformations are found in [Bell, 1994][1]. Another alternative is found in Appendiks A.

## 2.4. *Residual Bending Flexibility*

[H.MacNeal, 1978][3] introduce *Residual Bending Flexibility* in order to improve the two node element with one point Gauss integration. Using $G\bar{A}^s_\alpha$ instead of $GA^s_\alpha$ where

$$G\bar{A}^s_\alpha = \left( \frac{1}{GA^s_\alpha} + \frac{h^2}{12EI_\alpha} \right)^{-1}$$

result in an exact stiffness matrix well known from beam theory, when $GA^s_\alpha \to \infty$. The expression

$$\frac{h^2}{12EI_\alpha}$$

[1] Kolbein Bell. *Matrisestatikk.* Number ISBN: 82-519-1162-1 (ib.). Tapir, 1994.

[3] R. H.MacNeal. A simple quadrilateral shell element. *Computer & Structurer*, 8:175—183, 1978.

is termed *Residual Bending Flexibility*. Using one point integration together with residual bending flexibility the accuracy using linear basis function are then comparable to the one obtained with qubic functions.

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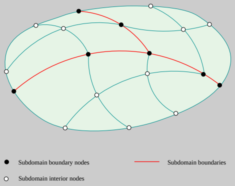


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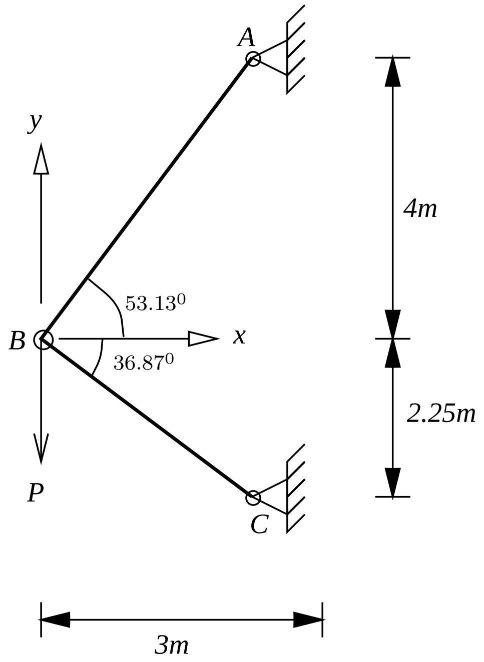
## 2.5. Fagverk (LINK1)




**Problem:** Fagverket $ABC$ som vises i figuren belastes med lasten $P = 3MN$. Stavene $AB$ og $BC$ har hhv. areal tverrsnitt $A_{ab} = 0.3m^2$ og $A_{bc} = 0.9m^2$. $E$-modulen til materialet (for

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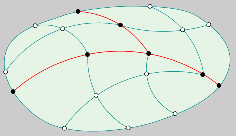


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begge stavene) er $E = 70 \times 10^9 N/m^2$.

a) Finn den horisontale og den vertikale forskyvningen av punkt $B$.

b) Hvor mye forlenges (forkortes) stavene?

c) Finn også spenningene i stavene.

d) Kan stavene inndeles i mer enn ett element?

**Løsning:** Input fil til ANSYS for dette eksempelet:

```
/BATCH,LIST
/FILNAM,ex411
/TITLE, Lineær statisk analyse av et fagverk

/PREP7
ET,1,1                 ! LINK1 elementer
R,1,0.3                ! Tverrsnitts areal til stav AB
R,2,0.9                !   og stav AC
MP,EX,1,70e9           ! E-modulen

!Geometri ("solid modelling")
K,1,0,0                ! Punkt B er origo
K,2,3,4                ! Punkt A
K,3,3,-2.25            ! Punkt C
L,1,2                  ! Linje AB
L,1,3                  ! og BC

!Inndeling i elementer
LESIZE,1,,,1           ! Deklarer at linjene 1 og 2 skal
```

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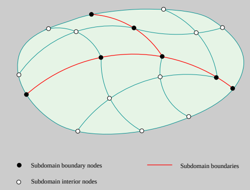


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```
LESIZE,2,,,1         ! inndeles i ett element.
REAL,1               ! Bruk tverrsnittsareal nr. 1 for inndelingen (neste linje)
LMESH,1              ! Inndeling av linje 1
REAL,2               ! Bruk tverrsnittsareal areal nr. 2 .....
LMESH,2
FINISH               ! Ut av Preprossessoren

/SOLU                ! Løsningsprossessoren
ANTYPE, STATIC       ! Statisk analyse (default)
DK,2,all             ! Null forskyvning for punkt A
DK,3,all             !  og C (alle)
FK,1,fy,-3e6         ! Belastning i punkt B
DTRAN                ! Overfører grensebetetingelser til elementmodell
SBCTRAN              ! og belastningen
SOLVE                ! Løsningsprosedyren
FINISH               ! Ut av Løsningsprossessoren

/POST1               ! Postprossessoren
SET                  ! Last inn analyseresultatene
PLDISP,1             ! Deformert konstruksjon
PRNSOL,U,COMP        ! Utskrift av forskyvningene (global akse)
LOCAL,11,0,,,,53.1301  ! Lokalt aksesystem
RSYS,11                ! aktiveres og brukes til å lese
PRNSOL,U,COMP          ! forskyvningene og
PRESOL, F              ! kreftene
PRESOL,ELEM            ! Ta ut tilgjengelige elementresultater (aksialkrefter)
FINISH
```

### Kommentarer:

- Et lokalt koordinatsystem med origo ved punkt $B$ og med akser parallelle med stavene brukes for beregning av stavenes spenninger og tøyninger. Koordinatsystemet defineres

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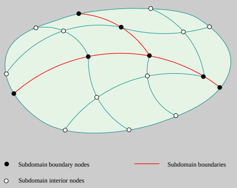




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med kommandoen `LOCAL` og aktiveres i postprosessoren med kommandoen `RSYS`. Aksiallastene i stavene kan også finnes ved kommandoen `PRESOL, ELEM`.

### Svar på spørsmålene:

a) Horisontal og vertikal forskyvningen av punkt B er gitt av

| Punkt | u | v |
|---|---|---|
| B | $-\frac{9}{35000}m$ | $-\frac{73}{140000}$ |
| | $-0.257143mm$ | $-0.521429mm$ |

b) Stav $AB$ forlenges $0.571429mm$ og stav $BC$ forkortes med $0.107143mm$.

c) Speningene i $AB$ og $BC$ er hhv. $8 \times 10^6 N/m^2$ (strekk) og $-2 \times 10^6 N/m^2$ (trykk).

d) Nei, da får man mekanismer.

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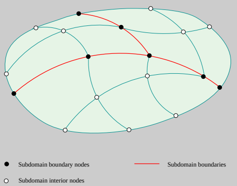


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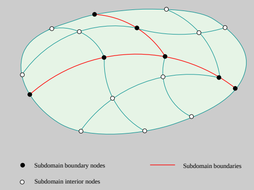
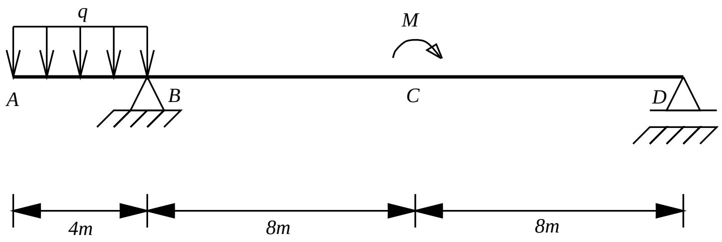





## 2.6. Bjelke (BEAM3)




$E = 2 \times 10^{11} N/m^2$

$I = 1.936 \times 10^{-3} m^4$

$M = 12 \times 10^{6} Nm$

$q = 150kN/m$ og $1500kN/m$

**Problem:** Den fritt opplagte bjelken $ABCD$ er belastet som vist i figuren.

a) Tegn opp skjær- og momentfordelingene ut fra ANSYS resultatene.

b) Hva er den vertikale forskyvningen av punktene $A$ og $C$? Bekreft dette med håndberegninger.

**Løsning:**

```
/BATCH,LIST
/FILNAM,ex412
/TITLE, Lineær statisk analyse av en bjelke

/PREP7                  ! Preprossessoren
ET,1,3                  ! BEAM3 elementer
```

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```
R,1,500,1.936e9,6.5e3  ! Høyde, I-verdi og areal til tverrsnitt
MP,EX,1,2e5            ! E-modulen

!Geometri ("solid modelling")
K,1,0                 ! Punkt A (er origo)
K,2,4e3               !       B
K,3,12e3              !       C
K,4,20e3              !  og   D
L,1,2                 ! Linje AB
L,2,3                 !       BC
L,3,4                 !   og  CD

!Inndeling i elementer
LSEL,S,line,,2,3        ! Velger linjene BC og CD
LESIZE,all,,,8          !  som inndeles i 8 elementer hver
LSEL,inve               ! Velger invers av det valgte settet (dvs. linje AB)
LESIZE,all,,,4          !  som deles inn i 4 elementer
LSEL,all                ! Velger all linjer igjen
REAL,1 $ TYPE,1 $ MAT,1 ! egentlig ikke nødvendig (de er default)
LMESH,1,3               ! Inndeling av alle tre linjer
FINISH                  ! Ut av Preprossessoren

/SOLU                       ! Løsningsprosessoren
ANTYPE,STATIC               ! Statisk analyse (default)
D,node(4e3,0,0),ux,,,,,uy   ! Opplagring ved B
D,node(20e3,0,0),uy         !    og         D
ESEL,s,elem,,1,4            ! Velger elementene mellom A og B
SFBEAM,all,1,pres,150e1     ! Den jevnt fordelte lasten (se elementets
                            ! beskrivelse for forklaringen av 2. argumentet)
!(PSF,PRES,NORM,2 og EPLO for å se på lasten)
ESEL,all
F,node(12e3,0,0),mz,-12e9   ! Momentet på C
SOLVE                       ! Løsningsprosedyren
FINISH                      ! Ut av løsningsprosessoren

/POST1                  ! Postprossessoren
SET                     ! Last inn kjøringsresultatene
```

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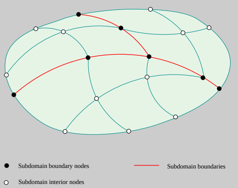


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```
/DSCALE,1,1              ! Virkelig skalering på deformert geometri
PLDISP,1                 ! Deformert konstruksjon
PRESOL,f                 !  .. gir skjaerkreftene
PRESOL,m                 !  .. gir momentene
ETABLE,IMOM,SMISC,6      ! moment på elementets ende i
ETABLE,JMOM,SMISC,12     !              ..........  j
ETABLE,ISKJ,SMISC,2      ! skjær på ende i
ETABLE,JSKJ,SMISC,8      !          .... j
/PLOPTS,INFO, OFF
/PLOPTS,TITLE,OFF
/SHOW,ex412pl            ! Plot til fil: ex412pl
PLLS,IMOM,JMOM           ! M-diagram
PLLS,ISKJ,JSKJ           ! V-disgram
FINISH
```

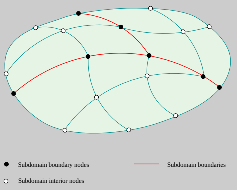
## Kommentarer:

- SELECT logikk i ANSYS demonstreres her. Dette er et kraftig verktøy som gjør manuelle repetisjon av kommandoer overflødig og er derfor sterkt anbefalt.
- I løsningsprossessoren er elementmodellen brukt som referanse når opplagringen defineres. På denne måten trenger man ikke å vite rekkefølgen og nummerne på knutepunktene eller "keypoints" (kun koordinatene til knutepunktene).

## Svar på spørsmålene:

a) Figuren nedenfor viser $M$- og $V$-diagrammer for bjelken for de to lasttilfellene. Rette



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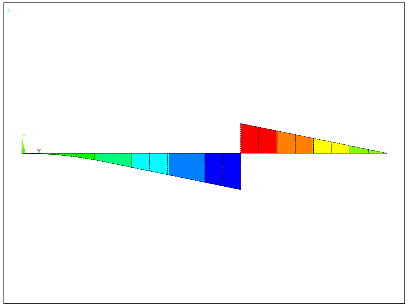
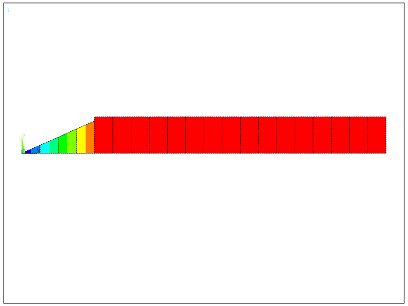
linjer forbinder knutepunktsverdiene. Med finere elementinndeling blir diagrammene mer lik konvensjonelle $M$- og $V$-diagrammer.

b) Nedbøyning av punktene $A$ og $C$ (`PRNSOL,U,COMP`) er hhv. $-161.16\,mm$ og $49.587\,mm$ for $q = 150\,kN/m$ og $867.77\,mm$ og $495.87\,mm$ for $q = 1500\,kN/m$.

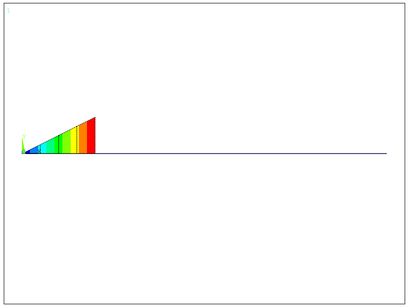
Moment $(q = 150\,kN/m)$



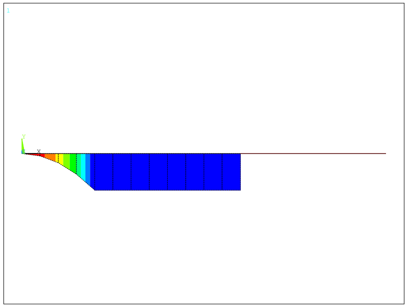
Skjærkraft $(q = 150\,kN/m)$



Moment $(q = 1500\,kN/m)$

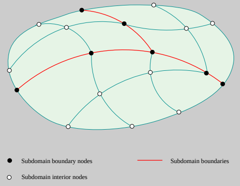


Skjærkraft $(q = 1500\,kN/m)$



Moment- og skjærkraft diagrammene for de to lasttilfellene





## Øving 2.1

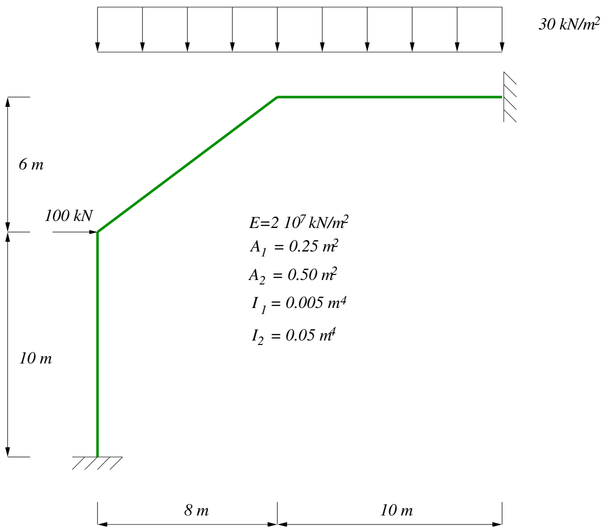
Benytt et to noders bjelkeelement basert på Mindlin-Reissner bjelketeori.

a) Beregn konsistent lastvektor for konstant tverrlast (2D).

b) Hvordan sammenligner dette med resultatene fra den klassiske bjelkeligningen?

## Øving 2.2

I denne oppgaven skal vi igjen se på oppgaven i [Bell, 1994][1] kapittel 3.4.





[1] Kolbein Bell. *Matrisestatikk*. Number ISBN: 82-519-1162-1 (ib.). Tapir, 1994.



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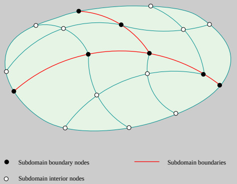



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Benytt bjelkelementet `BEAM3` i ANSYS.

a) Modeller konstruksjonen i ANSYS (geometri, materialdata, tverrsnittsdata og randkrav).

b) Sett på knutepunktslaster i henhold til [Bell, 1994][1] og foreta en analyse av problemet.

c) Tegn aksial-, skjær-, og momentdiagrammer. Hvordan sammenligner dette med [Bell, 1994][1]?

d) Sett på fordelte laster slik som gitt i figuren. Foreta en ny analyse.

e) Tegn aksial-, skjær-, og momentdiagrammer. Hvordan sammenligner dette med [Bell, 1994][1]?

f) Se bort fra skjær- og aksialdeformasjoner og foreta en ny analyse.

g) Tegn aksial-, skjær-, og momentdiagrammer. Hvordan sammenligner dette med [Bell, 1994][1]?

Benytt elementet `BEAM188`(Dette er er 3D element).

h) Sett på knutepunktslaster i henhold til 2.1 og foreta en analyse av problemet.

i) Sett på fordelte laster slik som gitt i figuren. Foreta en ny analyse.

j) Tegn aksial-, skjær-, og momentdiagrammer. Hvordan sammenligner dette med `BEAM3`?

[1] Kolbein Bell. *Matrisestatikk*. Number ISBN: 82-519-1162-1 (ib.). Tapir, 1994.

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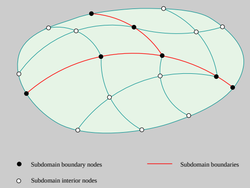


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# A. Alternative skjærformulering, *Timoshenko* bjelke

I dette kapittelet skal vi se på en alternative skjærformulering, en såkalt *Timoshenko* bjelke. Vi gjør dette i to dimensjoner. Dette er en korreksjon av *Euler-Bernoulli* bjelkeformuleringen som vi utviklet i *MEK4550, The Finite Element Method in Solid Mechanics I*, Kapittel 7.

Fra teknisk bjelkteori kjenner vi sammenhengen mellom skjærtøyning og skjærkraft:

$$\gamma_{xz} = \frac{\sigma_{xz}}{G} = \frac{Q_z}{GA_z^s} = w_0' + \theta_y$$

Videre kan vi benytte sammenhengen mellom skjærkraft og bøyemoment

$$Q_z = M_y' = -\left(EI_z w_0''\right)' = -EI_z w_0'''$$

dersom $EI_z$ er konstant som gir oss følgende uttrykk for rotasjonen

$$\theta_y = -\frac{EI_z w_0'''}{GA_z^s} - w_0'$$

Vi kan nå etablere stivhetsmatrisen fra uttrykket for tøyningsenergien

$$\mathcal{U}(w_0) = \frac{1}{2}\int_\ell \left(EI_z\kappa^2 + GA_z^s\gamma_{xz}^2\right)\,dx = \frac{1}{2}\int_\ell \left(EI_z(w_0'')^2 + \frac{EI_z^2}{GA_z^s}(w_0''')^2\right)\,dx$$

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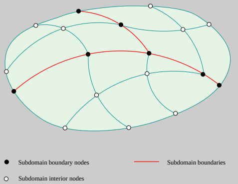

Interpolasjonspolynomene er fortsatt kubiske med $(w_i, \theta_{yi})$ som frihetsgrader i de to nodene.

$$w_0 = \begin{Bmatrix} 1 & x & x^2 & x^3 \end{Bmatrix} \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \boldsymbol{N}_q \boldsymbol{q}$$

Først finner vi uttrykket for den generaliserte stivhetsmatrisen. Stivhetsmatrisen er gitt ved

$$\boldsymbol{k}_q = \int_\ell \left( EI_z (\boldsymbol{N}_q'')^T \boldsymbol{N}_q'' + \frac{EI_z^2}{GA_z^s} (\boldsymbol{N}_q''')^T \boldsymbol{N}_q''' \right) dx$$

hvor

$$\boldsymbol{N}_q'' = \begin{Bmatrix} 0 & 0 & 2 & 6x \end{Bmatrix} \qquad \text{og} \qquad \boldsymbol{N}_q''' = \begin{Bmatrix} 0 & 0 & 0 & 6 \end{Bmatrix}$$

Dette gir

$$\boldsymbol{k}_q = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4\ell & 6\ell^2 \\ 0 & 0 & 6\ell^2 & 12\left(\ell^3 + 3\frac{EI\ell}{GA_z^s}\right) \end{bmatrix}$$

Vi innfører

$$\alpha = \frac{12EI_z}{GA_z^s \ell^2} \qquad \rightarrow \qquad \boldsymbol{k}_q = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4\ell & 6\ell^2 \\ 0 & 0 & 6\ell^2 & 3\ell^3 (4 + \alpha) \end{bmatrix}$$

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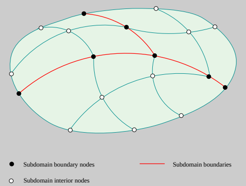




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For å komme fram til stivhetsmatrisen relatert til elementets nodefrihetsgrader trenger vi relasjonen mellom generaliserte koordinater og nodeforskyvninger

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -\frac{\alpha\ell^2}{2} \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & -1 & -2\ell & -\frac{\alpha\ell^2}{2} - 3\ell^2 \end{bmatrix}$$

Stivhetsmatrisen kan nå uttrykkes som

$$\boldsymbol{k} = \boldsymbol{A}^{-T}\boldsymbol{k}_q\boldsymbol{A}^{-1} = \frac{EI_z}{(1+\alpha)\ell^3}\begin{bmatrix} 12 & -6\ell & -12 & -6\ell \\ -6\ell & \ell^2(\alpha+4) & 6\ell & \ell^2(2-\alpha) \\ -12 & 6\ell & 12 & 6\ell \\ -6\ell & \ell^2(2-\alpha) & 6\ell & \ell^2(\alpha+4) \end{bmatrix}$$

**Remark A.1** En annen tilnærming til avanserte en dimensjonal elementer kan finnes i Introduction to Finite Element Methods (ASEN 5007)

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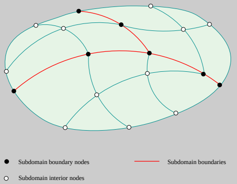


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# B. References


[Bell, 1994] Bell, K. (1994). *Matrisestatikk*. Number ISBN: 82-519-1162-1 (ib.). Tapir.

[Cook et al., 2002] Cook, R. D., Malkus, D. S., Plesha, M. E., and Witt, R. J. (2002). *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition.

[H.MacNeal, 1978] H.MacNeal, R. (1978). A simple quadrilateral shell element. *Computer & Structurer*, 8:175—183.

[Hughes, 1987] Hughes, T. J. R. (1987). *The Finite Element Method, Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey.



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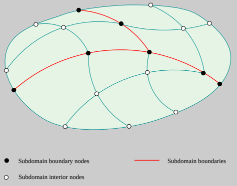



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