Chapter: 5

MEK4560
The Finite Element Method in Solid Mechanics II

(February 11, 2008)

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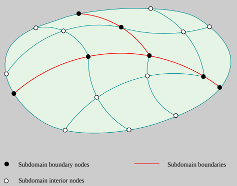

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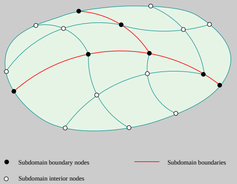

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5. Thick plates

In this Chapter we derive model for thick plates, called *Mindlin-Reissner* plates, following the same path as for beams and thin plates. In Chapter Chapter 4 we derived a model for thin plates and we recall that the potential energy functional had terms with second order derivatives, thus a conforming finite element method require C^1 continuity across inter element boundaries.

The Mindlin-Reissner model for thick plates involves only second derivatives in the potential energy functional, consequently C^0 continuity is sufficient in a conforming finite element method. However, *shear locking* is a problem for sufficiently thin plates, similar to the Mindlin-Reissner beam.

The material on plates is found in [Cook et al., 2002][1] Chapters 15.1, 15.2, 15.3, 15.4 and 15.5. The derivation follow [Hughes, 1987][2].

[1] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition, October 2002.

[2] T. J. R. Hughes. *The Finite Element Method, Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

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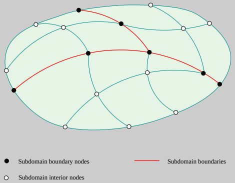

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5.1. Assumptions

The thick plate theory is based on the following assumptions:

1. The geometry;

$$V = \left\{(x,y,x) \in \mathbb{R}^3 | (x,y) \in \mathcal{A} \in \mathbb{R}^2,\ z \in [-\frac{t}{2}, \frac{t}{2}]\right\}$$

   where t is the thickness of the plate and $\mathcal{A}$ is the middle plane.

2. Plane stress, $\sigma_{zz} = 0$.
3. Small displacements and rotations:

$$w(x,y) << t, \qquad \sin\alpha = \alpha = -\theta_y$$

   and similarly for rotations around the x-axis.

4. The displacements in the plane is given by:

$$u(x,y,z) = z\theta_y(x,y) \qquad \text{and} \qquad v(x,y,z) = -z\theta_x(x,y) \tag{5.1}$$

   where two new parameters are introduced in order to model the rotation of a fiber[1]

5. The out of plane displacements are independent of z, $w(x,y,z) = w(x,y)$.

[1] The line defined by fixing (x,y) in equation (5.1) is a *fiber*, while the plane defined by fixing z is a *lamina*.

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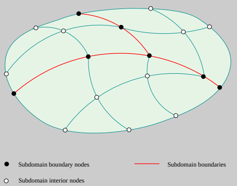

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Remark 5.1 One could also add axial displacements in the x and y directions. Since we use the mid plane in the model, the axial terms are independent of the bending terms. We therefor concentrate on the bending part in our derivation.

Remark 5.2 In general the thickness may be a function of x and y, $t = t(x, y)$. Again, the variation should not introduce three dimensional stress effects.

Remark 5.3 Other definitions of positive angles are used by some authors. In [Hughes, 1987][2] the sign of the angles are chosen to be able to use an index form of tensor notation.

[2] T. J. R. Hughes. *The Finite Element Method, Linear Static and Dynamic Finite Element Analysis.* Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

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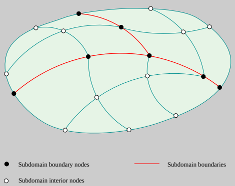

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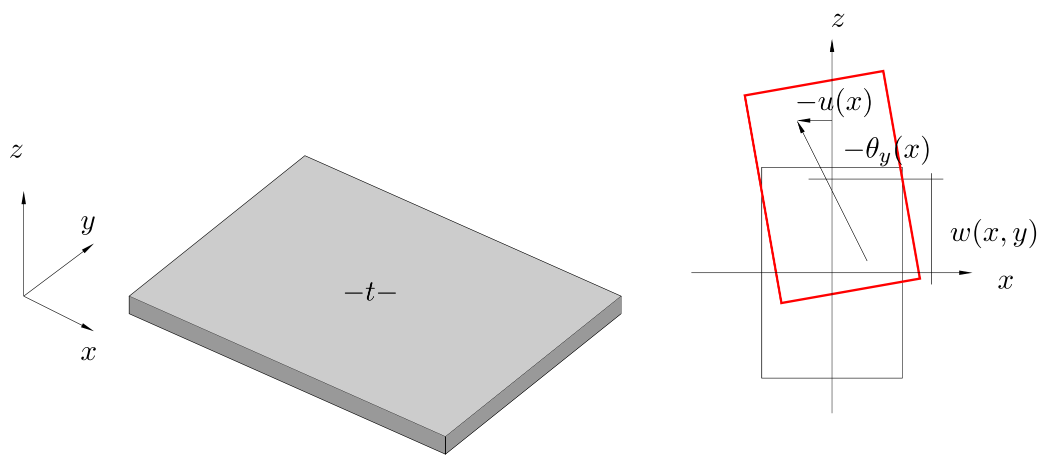

Figure 5.1: Plate geometry. Deformation of a plate in a y cross section.

Item 2 and 4 above are inconsistent, however experience show that the model is useful in practice. Since vi har innført muligheten for at platefibrene ikke trenger å være normalt på referanseplanet har vi skjærtøyninger på tvers.

5.2. Kinematics

Displacements: Above the displacements where defined as

$$u(x,y,z) = z\theta_y(x,y), \qquad v(x,y,z) = -z\theta_x(x,y) \qquad \text{and} \qquad w(x,y,z) = w(x,y)$$

see also Figure 5.1 This can be summarized as:

> *Plane cross sections remains plane after deformation, but not necessarily normal to the reference plane.*

Using the displacement field it is straightforward to find:

- strains, stresses, equilibrium, and the potential energy functional.

Strains: The strains are out of plane:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ w_{,x} + u_{,z} \\ w_{,y} + v_{,z} \end{pmatrix} = \begin{pmatrix} z\theta_{y,x} \\ -z\theta_{x,y} \\ -z\theta_{x,x} + z\theta_{y,y} \\ \theta_y + w_{,x} \\ -\theta_x + w_{,y} \end{pmatrix}$$

It is convenient to split the strain vector in two parts:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \boldsymbol{\varepsilon}_b \\ \boldsymbol{\varepsilon}_s \end{pmatrix}$$

where $\boldsymbol{\varepsilon}_b$ is the bending part:

$$\boldsymbol{\varepsilon}_b = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = -z \begin{pmatrix} -\theta_{y,x} \\ \theta_{x,y} \\ \theta_{x,x} - \theta_{y,y} \end{pmatrix} = -z\boldsymbol{\kappa}$$

and $\boldsymbol{\varepsilon}_s$ is the shear part:

$$\boldsymbol{\varepsilon}_s = \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} \theta_y + w_{,x} \\ -\theta_x + w_{,y} \end{pmatrix}$$

Here $\gamma_{\alpha z}$ is the difference between the normal fiber and the material fiber. .

Remark 5.4 The plate theaory also has microscopic inconsistencies. The normal strains, ε_{zz}, are zero and this indicate a plane strain condition. However, the assumption $\sigma_{zz} = 0$ is more physical. For an isotropic material this is possible only when $\nu = 0$.

5.3. The material law

The relations between stress and strains are from the relation for linearized elasticity:

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}$$

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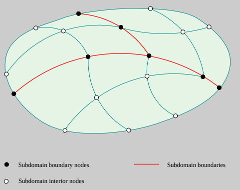

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For thick plates it is convenient to split it into two parts:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_b \\ \boldsymbol{\sigma}_s \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}_b & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}_s \end{pmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_b \\ \boldsymbol{\varepsilon}_s \end{pmatrix} \quad \text{where} \quad \boldsymbol{\sigma}_b = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\sigma}_s = \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix}$$

The stresses are integrated across the thickness to obtain moments and shear forces:

$$\begin{pmatrix} \boldsymbol{M} \\ \boldsymbol{Q} \end{pmatrix} = \int_{-\frac{t}{2}}^{-\frac{t}{2}} \begin{pmatrix} -z\boldsymbol{\sigma}_b \\ \boldsymbol{\sigma}_s \end{pmatrix} dz = \begin{pmatrix} \boldsymbol{D}_b & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_s \end{pmatrix} \begin{pmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\varepsilon}_s \end{pmatrix} \tag{5.2}$$

where

$$\boldsymbol{D}_b = \int_{-\frac{t}{2}}^{-\frac{t}{2}} z^2 \boldsymbol{E}_b \, dz \quad \text{and} \quad \boldsymbol{D}_s = \int_{-\frac{t}{2}}^{-\frac{t}{2}} \boldsymbol{E}_s \, dz$$

Note that if $\boldsymbol{E}_b$ and $\boldsymbol{E}_s$ are constant across the plate

$$\boldsymbol{D}_b = \frac{t^3}{12} \boldsymbol{E}_b \quad \text{and} \quad \boldsymbol{D}_s = t\boldsymbol{E}_s \tag{5.3}$$

Remark 5.5 To be consistent with the classical theory a shear correction factor κ may be introduced, and $\bar{\boldsymbol{D}}_s = \kappa \boldsymbol{D}_s$. For plates the correction factor is $\kappa = \frac{5}{6}$.

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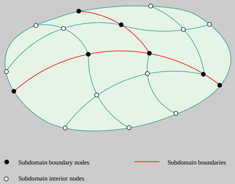

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5.4. Potential energy

The potential energy is found from the general expression for strain energy, where the internal strain energy is written as a sum of bending and shear energy:

$$\begin{aligned}\Pi(w,\theta_x,\theta_y) &= U_b(\boldsymbol{\kappa}) + U_s(\boldsymbol{\varepsilon}_s) - W \\ &= \frac{1}{2}\int_A \boldsymbol{\kappa}^T \boldsymbol{D}_b \boldsymbol{\kappa} + \boldsymbol{\varepsilon}_s^T \boldsymbol{D}_s \boldsymbol{\varepsilon}_s \, dA - \int_A q w \, dA\end{aligned}$$

Note that if the shear strains are zero, i.e. $w_{,x} - \theta_y = 0$ and $w_{,y} + \theta_x = 0$, the energy functional is equal to the functional for thin plates.

Remark 5.6 The shear strains measure the differences in angles between a plane normal to the axis after deformation and the actual plane given by θ_x and θ_y

Remark 5.7 Using (5.3) the potential energy functional becomes

$$\Pi(w,\theta_x,\theta_y) = \frac{1}{2}\int_A \frac{t^3}{12}\boldsymbol{\kappa}^T \boldsymbol{E}_b \boldsymbol{\kappa} + t\boldsymbol{\varepsilon}_s^T \boldsymbol{E}_s \boldsymbol{\varepsilon}_s \, dA - \int_A q w \, dA \tag{5.4}$$

Thus, when t becomes small the shear term dominates the equation and shear locking occurs.

Remark 5.8 The potential energy functional is not complete, a number of boundary terms may be added to take non homogeneous boundary conditions into account, see Hughes [Hughes, 1987]. For some terms see subsection 5.7.

Remark 5.9 In order to find the correct definition of signs of the angles and definition of the boundary condition, refer to the manual of the FEM program.

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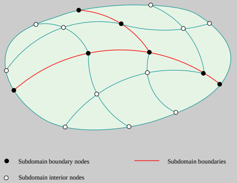

5.5. Equilibrium equations I

Recall from Chapter 4 that the functions w and $\boldsymbol{\theta}$ minimizing the potential energy functional satisfies

$$\int_A \boldsymbol{\kappa}^T(\boldsymbol{\phi})\boldsymbol{D}_b\boldsymbol{\kappa}(\boldsymbol{\theta}) + \boldsymbol{\varepsilon}_s^T(\boldsymbol{\phi}, v)\boldsymbol{D}_s\boldsymbol{\varepsilon}_s(\boldsymbol{\theta}, w)\, dA = \int_A qv\, dA \quad \text{for all } \boldsymbol{\phi} \text{ and } v \tag{5.5}$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ are vector functions

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_x \\ \theta_y \end{pmatrix} \quad \text{and} \quad \boldsymbol{\phi} = \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix} \tag{5.6}$$

In order to derive the equilibrium equations from the weak form Greens formula is used. It is convenient to use the moments and shear forces in the weak formulation

$$\int_A \boldsymbol{\kappa}^T(\boldsymbol{\phi})\boldsymbol{M}(\boldsymbol{\theta}) + \boldsymbol{\varepsilon}_s^T(\boldsymbol{\phi}, v)\boldsymbol{Q}(\boldsymbol{\theta}, w)\, dA = \int_A qv\, dA \quad \text{for all } \boldsymbol{\phi} \text{ and } v \tag{5.7}$$

Here

$$\boldsymbol{\kappa}^T(\boldsymbol{\phi})\boldsymbol{M} = -\phi_{y,x}M_{xx} + \phi_{x,y}M_{yy} + (\phi_{x,x} - \phi_{y,y})M_{xy} \tag{5.8}$$

and

$$\boldsymbol{\varepsilon}_s^T(\boldsymbol{\phi}, v)\boldsymbol{Q} = \phi_y Q_{xz} + v_{,x}Q_{xz} - \phi_x Q_{yz} + v_{,y}Q_{uz} \tag{5.9}$$

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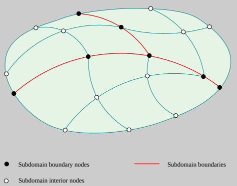

Greens formula is used one each moment term

$$\int_A \phi_{y,x} M_{xx} \, dA = \int_{\partial A} \phi_y M_{xx} n_x \, dS - \int_A \phi_y M_{xx,x} \, dA \tag{5.10}$$

$$\int_A \phi_{x,y} M_{yy} \, dA = \int_{\partial A} \phi_x M_{yy} n_y \, dS - \int_A \phi_x M_{yy,y} \, dA \tag{5.11}$$

$$\int_A \phi_{x,x} M_{xy} \, dA = \int_{\partial A} \phi_x M_{xy} n_x \, dS - \int_A \phi_x M_{xy,x} \, dA \tag{5.12}$$

$$\int_A \phi_{y,y} M_{xy} \, dA = \int_{\partial A} \phi_y M_{xy} n_y \, dS - \int_A \phi_y M_{xy,y} \, dA \tag{5.13}$$

$$\tag{5.14}$$

and on each force therm with force multiplied with the derivative of the basis function v

$$\int_A v_{,x} Q_{xz} \, dA = \int_{\partial A} v \, Q_{xz} n_x \, dS - \int_A v \, Q_{xz,x} \, dA \tag{5.15}$$

$$\int_A v_{,y} Q_{yz} \, dA = \int_{\partial A} v \, Q_{yz} n_y \, dS - \int_A v \, Q_{yz,y} \, dA \tag{5.16}$$

$$\tag{5.17}$$

Collecting terms we obtain

$$\begin{aligned} &\int_A \phi_y (M_{xx,x} + M_{xy,y} + Q_{xz}) - \phi_x (M_{yy,y} + M_{xy,x} + Q_{yz}) - v (Q_{xz,x} + Q_{yz,y}) \, dA \\ &+ \int_{\partial A} -\phi_y (M_{xx} n_x + M_{xy} n_y) + \phi_x (M_{yy} n_y + M_{xy} n_x) + v (Q_{xz} n_x + Q_{yz} n_y) \, dS = \int_A q \, v \, dA \end{aligned}$$

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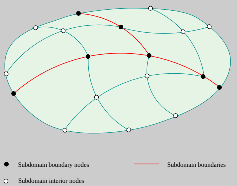

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Setting $\phi_x = \phi_y = 0$ and $v = 0$ on the boundary, it follows that

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} = q \tag{5.18}$$

in the mid plane of the plate.

Setting $\phi_x = 0$ and $\phi_y = v = 0$ on the boundary, it follows that

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_{xz} = 0 \tag{5.19}$$

in the mid plane of the plate.

Setting $\phi_x = \phi_y = v = 0$ on the boundary, it follows that

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + Q_{yz} = 0 \tag{5.20}$$

in the mid plane of the plate.

Substituting the expressions (5.2) into equations (5.18), (5.19) and (5.20) we obtain a set of three coupled second order partial differential equations. In addition suitable boundary conditions are required.

5.6. Boundary conditions

In order to proceed, we set $\phi_x = \phi_y = 0$ on the boundary. Then

$$Q_{xz}n_x + Q_{yz}n_y = Q_n = 0 \tag{5.21}$$

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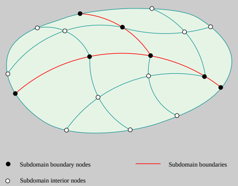

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on (a part) of the boundary

The last term we rewrite using the normal and tangential part of the vector function $(-\phi_y, \phi_x)$, i.e.

$$\begin{pmatrix} -\phi_y \\ \phi_x \end{pmatrix} = n^T \begin{pmatrix} -\phi_y \\ \phi_x \end{pmatrix} n + s^T \begin{pmatrix} -\phi_y \\ \phi_x \end{pmatrix} s = \boldsymbol{\phi}_s n + \boldsymbol{\phi}_n s \tag{5.22}$$

Using this

$$\begin{aligned} &-\phi_y(M_{xx}n_x + M_{xy}n_y) + \phi_x(M_{xy}n_x + M_{yy}n_y) \\ &\qquad = (-\phi_y\ \phi_x) \begin{pmatrix} M_{xx}n_x + M_{xy}n_y \\ M_{xy}n_x + M_{yy}n_y \end{pmatrix} = \boldsymbol{\phi}_s M_{nn} + \boldsymbol{\phi}_n M_{ns} = 0 \end{aligned} \tag{5.23}$$

The homogeneous boundary conditions are summarized in the table below:

Condition	A	B	C
Clamped	$w = 0$	$\theta_n = 0$	$\theta_s = 0$
Free	$Q_n = 0$	$M_{ns} = 0$	$M_{nn} = 0$
Simply supported, hard	$w = 0$	$\theta_n = 0$	$M_{nn} = 0$
Simply supported, soft	$w = 0$	$M_{ns} = 0$	$M_{nn} = 0$
Symmetric around s	$Q_n = 0$	$M_{ns} = 0$	$\theta_s = 0$
Antisymmetric around s	$w = 0$	$\theta_n = 0$	$M_{nn} = 0$

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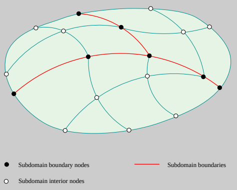

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Note that the clamped boundary conditions are essential conditions and is usually enforced in the finite element space, while the free conditions are natural conditions and are part of the functional if they are non-homogeneous. The others are a mixture of essential and natural conditions.

In order to incorporate non-homogeneous conditions the potential energy functional can be taken to be

$$\Pi(w, \theta_x, \theta_y) = \frac{1}{2} \int_A \boldsymbol{\kappa}^T \boldsymbol{D}_b \boldsymbol{\kappa} + \boldsymbol{\varepsilon}_s^T \boldsymbol{D}_s \boldsymbol{\varepsilon}_s \, dA - \int_A qw \, dA - \int_S \left[Q_n w + M_{nn}\theta_s + M_{ns}\theta_n \right] \, ds$$

where Q_n, M_{nn} and M_{ns} are given quantities on the boundary. The boundary can be divided in different pieces

$$S = S_w \cup S_Q \qquad \text{where} \qquad S_w \cap S_Q = \emptyset$$

Similarly

$$S = S_{\theta_n} \cup S_{M_{ns}} \qquad \text{where} \qquad S_{\theta_n} \cap S_{M_{ns}} = \emptyset$$

and

$$S = S_{\theta_s} \cup S_{M_{nn}} \qquad \text{where} \qquad S_{\theta_s} \cap S_{M_{nn}} = \emptyset$$

Note that there are two possibilities for *Simply supported* conditions. In some cases the *hard* condition is to strong, the model predict a stiffer plate than the *soft* condition.

For thin plates there was corner forces, for Mindlin-Reissner plates a substantial increase in Q_n is observed for corners with soft conditions. This is more realistic than concentrated corner forces. The increase in forces is not observed with hard conditions.

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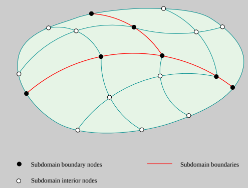


Not that the boundary conditions for thick plate theory are different from classical thin plate theory.

5.7. Equilibrium equations II

Since shear strains are included, the equilibrium equations can also be derived from the three dimensional model of solid mechanics, see *MEK4550*.

The equilibrium equations can be written

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \tag{5.24}$$

Integrating across the thickness in the x-direction of the equation becomes

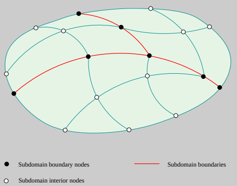
$$\int_{-\frac{t}{2}}^{-\frac{t}{2}} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x \, dz = 0$$

where the integral of σ_{xx} and σ_{xy} across the thickness is zero in our case, no axial forces. Similarly in the y-direction. Integration of the z equation result in

$$\int_{-\frac{t}{2}}^{-\frac{t}{2}} \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z \, dz = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z = 0$$

Here we have used Equation 5.2, $\sigma_{zz} = 0$ and

$$q_z = \int_{-\frac{t}{2}}^{-\frac{t}{2}} F_z \, dz$$



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In order to derive the momentum equations, multiply Equation 5.24 with $-z$:

$$\int_{-\frac{t}{2}}^{-\frac{t}{2}} -z\Big(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x\Big)\, dz = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x - m_x = 0 \qquad (5.25)$$

Here m_x is zero in our case. For the y-direction

$$\int_{-\frac{t}{2}}^{-\frac{t}{2}} -z\Big(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + F_y\Big)\, dz = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + Q_y - \underset{=0}{m_y} = 0$$

5.8. Summary

In summary:

Highest derivative m (in the PMPE)	1	
Highest derivative $2m$ (in the differential equation)	2	
Kinematic boundary conditions	0	(w, θ_n, θ_s)
Natural boundary conditions	1	(M_{nn}, M_{ns}, Q_n)
Continuity requirements $m-1$	0	(w, θ_x, θ_y)

Note, only C^0 continuity is required for a finite element method.

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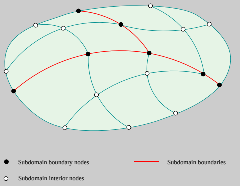

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Øving 5.1

Vis hvordan vi kommer fram til ligning Equation 5.25. Hva skjuler seg i m_x? (Anta at vi har kjente størrelser på flaten $S_t = (x, y, -t/2) \cup (x, y, t/2)$. Benytt ligningene fra elastisitetsteori.)

Øving 5.2

Varier tykkelsen på platen gitt i Øving 4.2. Benytt ANSYS eller *COMSOL Multiphysics* til å finne ut når platen begynner å bli tykk. Benytte grenseverdiene gitt i Kapittel 4 som utganspunkt.

Hvordan er hjørnekreftene sammenlignet med tynnplateteori? Benytt både *harde* og *myke* randkrav.

Øving 5.3

Benytt ANSYS og løs Oppgave C15.2 i [Cook et al., 2002][1].

[1] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition, October 2002.

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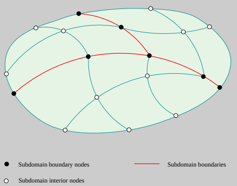

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A. References

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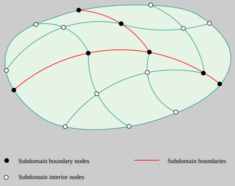

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