Chapter: 6

MEK4560
The Finite Element Method in Solid Mechanics II

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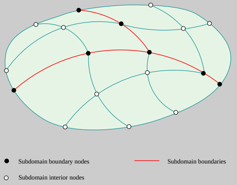

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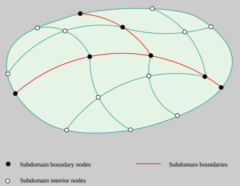

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6. Finite elements for plates

This chapter discuss finite elements for the plate model for thin, Chapter 4, and thick, Chapter 5, plates.

See C15.2 and C15.3 in [Cook et al., 2002][1].

6.1. Background

A number of methods used to establish plate elements are found in the literature:

1. Displacement methods:

Thin plates	C^1 continuity
	Variational crimes and the *Patch test*.
Thick plates	C^0 continuity
	Shear locking

2. Discrete Kirchhoff theory for thin plates. Introduce too many degrees of freedom, uses

[1] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition, October 2002.

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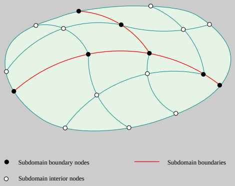

kinematics and zero shear conditions to reduce the number of degrees of freedom to three for each corner.

3. New element formulations, *Free Formulation* (FF), *Extended Free Formulation* (EFF), *Assumed Natural Strains* (ANS), *Assumed Natural Deviatoric Strains* (ANDES), *Enhanced Strains*, etc.

   Satisfy the patch test, Chapter 3.

4. Hybrid stress formulations.

6.2. Thin plates: Displacement formulation

The displacements can be represented in at least two ways:

1. Interpolation:

$$w = \boldsymbol{N}\boldsymbol{d}$$

   where $\boldsymbol{d}$ are generalized displacements at the nodal points. (Degrees of freedom can e.g. be rotations and bending.)

2. Generalized degrees of freedom:

$$w = \boldsymbol{N}_q\boldsymbol{q}$$

   where $\boldsymbol{q}$ are coefficients not necessarily related to nodal values.

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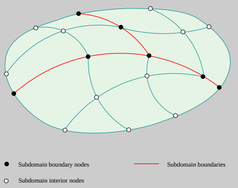

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For conforming elements both $\boldsymbol{N}$ and $\boldsymbol{N}_q$ are required to satisfy the *completeness* and the *compatibility* criteria:

1. Rigid body motion.
2. Constant strain.

$$\begin{array}{ccccc} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \end{array}$$

3. The displacements must be C^1-continuous, i.e. the function and both first order partial derivatives must be continuous across interelement boundaries.

   A finite element function is C^1 continuous if:

   - it is C^0 continuous.
   - the normal derivative $\frac{\partial w}{\partial n}$ is continuous across interelement boundaries.

Recall that the gradient along an edge can be split into a normal and a tangential derivative

$$\nabla w = (\nabla w)^T n\, n + (\nabla w)^T s\, s = \frac{\partial w}{\partial n} n + \frac{\partial w}{\partial s} s \tag{6.1}$$

If a polynomial finite element function w is continuous along an edge, the tangential derivative $\frac{\partial w}{\partial s}$ is also continuous, thus if the normal derivative $\frac{\partial w}{\partial n}$ continuous, the function is in C^1

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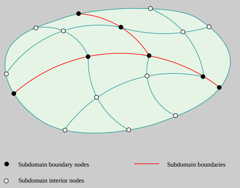

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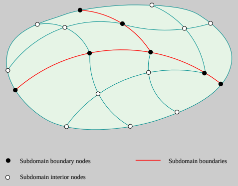
6.3. Conforming methods

We first consider some well known C^1 elements, both triangular and rectangular elements.

6.3.1. Triangular elements

Argyris triangle: ...the simplest element using polynomial basis functions achieving C^1 continuity. The basis functions are quintic polynomials

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & x & & y & & & & \\
 & & & x^2 & & xy & & y^2 & & & \\
 & & x^3 & & x^2y & & xy^2 & & y^3 & & \\
 & x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 & \\
x^5 & & x^4y & & x^3y^2 & & x^2y^3 & & xy^4 & & y^5
\end{array}
$$



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This i 21 degrees of freedom! The degrees of freedom are

$$\boldsymbol{d} = \begin{pmatrix} w \\ \theta_x \\ \theta_y \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{pmatrix}$$

at the corner points and

$$\boldsymbol{d}_s = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \qquad \text{where} \qquad \theta = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y$$

at the midpoints of the edges. This is $6 \cdot 3 + 3 = 21$ degrees of freedom. Note that θ_x and θ_y is the first order partial derivatives of w and that κ_{xx}, κ_{yy} and κ_{xy} are second order partial derivatives The element satisfies

1. C^1 continuity and completeness.
2. High order of interpolation result in high order of convergence, provided the solution is smooth.
3. Expensive if the solution is not smooth enough or the high accuracy is not required.

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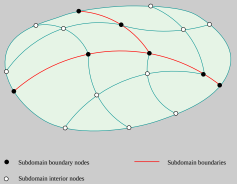

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4. Enforce continuity in bending, if material coefficients or thickness vary across interelement boundary they are not continuous.

Note: The degrees of freedom could be taken to be directional derivatives along the element edges instead of partial derivatives.

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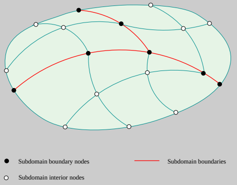

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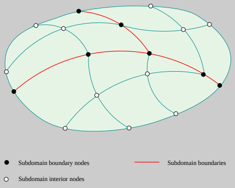
Bell triangle The Bell triangle is similar to the Argyris triangles except that the polynomials is required to be cubic along the edges. This element has 18 degrees of freedom, the corner point degrees of freedom used for the Argyris triangle. This is an advantage in many finite element codes that requires the degrees of freedom to be related to the corners. In addition the work is slightly reduced.



Bogner-Fox-Schmit rectangle The basis functions in this element are based on bicubic polynomials, i.e. tensor products of cubic polynomials. They have 16 degrees of freedom. At each node they are

$$\boldsymbol{d} = \begin{pmatrix} w \\ \theta_x \\ \theta_y \\ \kappa_{xy} \end{pmatrix}$$

This is a conforming element, i.e. C^1 continuous. The basis functions are

$$\begin{aligned}
\boldsymbol{N} = \Big\{ & \frac{1}{16}(\eta-1)^2(\eta+2)(\xi-1)^2(\xi+2), \frac{1}{32}b(\eta-1)^2(\eta+1)(\xi-1)^2(\xi+2), \\
& -\frac{1}{32}a(\eta-1)^2(\eta+2)(\xi-1)^2(\xi+1), \frac{1}{64}ab(\eta-1)^2(\eta+1)(\xi-1)^2(\xi+1), \\
& -\frac{1}{16}(\eta-1)^2(\eta+2)(\xi-2)(\xi+1)^2, -\frac{1}{32}b(\eta-1)^2(\eta+1)(\xi-2)(\xi+1)^2, \\
& -\frac{1}{32}a(\eta-1)^2(\eta+2)(\xi-1)(\xi+1)^2, \frac{1}{64}ab(\eta-1)^2(\eta+1)(\xi-1)(\xi+1)^2, \\
& -\frac{1}{16}(\eta-2)(\eta+1)^2(\xi-1)^2(\xi+2), \frac{1}{32}b(\eta-1)(\eta+1)^2(\xi-1)^2(\xi+2), \\
& \frac{1}{32}a(\eta-2)(\eta+1)^2(\xi-1)^2(\xi+1), \frac{1}{64}ab(\eta-1)(\eta+1)^2(\xi-1)^2(\xi+1), \\
& \frac{1}{16}(\eta-2)(\eta+1)^2(\xi-2)(\xi+1)^2, -\frac{1}{32}b(\eta-1)(\eta+1)^2(\xi-2)(\xi+1)^2, \\
& \frac{1}{32}a(\eta-2)(\eta+1)^2(\xi-1)(\xi+1)^2, \frac{1}{64}ab(\eta-1)(\eta+1)^2(\xi-1)(\xi+1)^2 \Big\}
\end{aligned}$$

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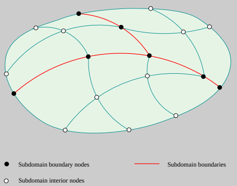

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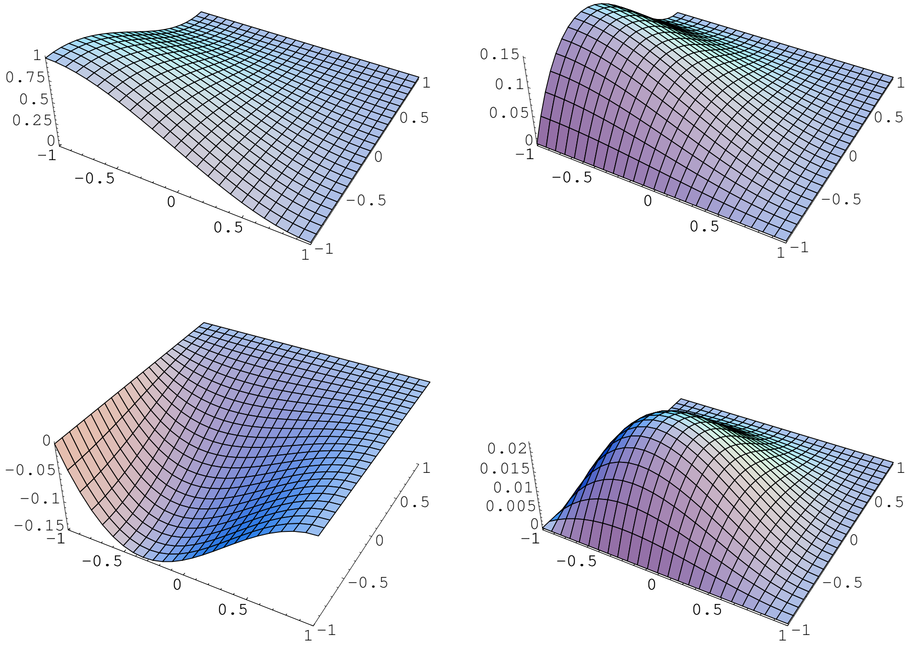
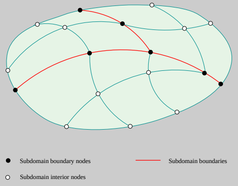
They are depicted in the figure below



This element has fewer degrees of freedom than the Argyris and Bell triangle, are less accurate



in theory, but in practice the analytical solution of the problems frequently are not sufficiently high to achieve the accuracy of the triangular elements. This is good choice of element for thin plate analysis.

6.4. Nonconforming methods

The elements introduced above are based on high degree polynomials and the computation of the element matrices are costly. Moreover, in many cases the solution of the plate problems are not sufficiently smooth to achieve the high accuracy of the elements. Thus finite element methods based on polynomials of lower degree could be more efficient.

To be able to reduce the order of the polynomials the C^1 continuity requirement must be relaxed. Usually the continuity of the function and the normal derivative is required at certain points. In this case the integrals in the potential energy functional, or the weak form, is well defined only in the interior of the elements. In practice the integrals are computed element by element, thus this is not a problem.

Morely triangle: The simplest polynomials allowing constant strains are quadratic polynomials. It has six degrees of freedom, the freedoms chosen for a forth order equation are

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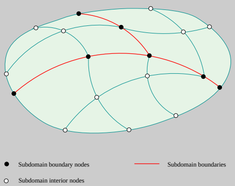

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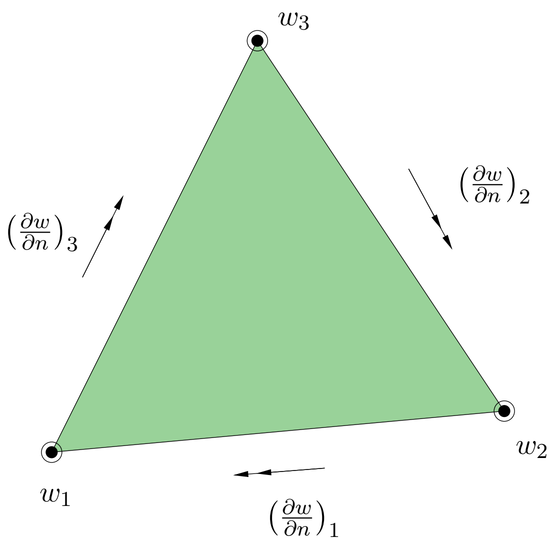

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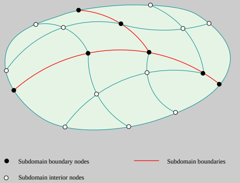

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i.e.

$$\boldsymbol{d} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \qquad \text{where} \qquad \theta = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y$$

The element do not satisfy C^1 continuity, the required continuity is only enforced at certain points.

The element

1. is complete.
2. is nonconforming.
3. satisfies the *Patch test*.
4. is similar to the CST element for membranes.
5. $w_{,n}$ is inconvenient as a degree of freedom in most finite element programs, $w^1_{,n} = -w^2_{,n}$.

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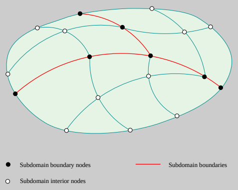

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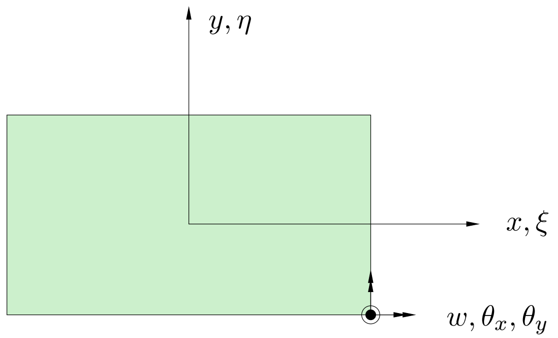
6.4.1. Rectangular elements

Adini-Clough-Melosh element: …the degrees of freedom are indicated on the figure below:



The degrees of freedom in each node are

$$\begin{pmatrix} w \\ \theta_x \\ \theta_y \end{pmatrix} \qquad \text{where} \qquad \theta_x = \frac{\partial w}{\partial y} \quad \text{and} \quad \theta_y = -\frac{\partial w}{\partial x}$$

Thus, the element has 12 degrees of freedom. The basis functions are polynomials composed

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of the terms (given by Adini/Clough)

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & x & & y & & \\
 & x^2 & & xy & & y^2 & \\
x^3 & & x^2y & & xy^2 & & y^3 \\
 & x^3y & & & & xy^3 &
\end{array}
$$

Is this a good element?

1. Straightforward to compute the basis functions $\boldsymbol{N}$.
2. Second degree polynomial, complete.
3. Non conforming. The element is C^0 continuous, since the functions are cubic along an edge and we have four degrees of freedom to specify the function. The normal derivatives are not continuous across inter element boundaries in general.
4. The element does not pass the *Patch test*.

Note: this element also has a triangular variant with nine degrees of freedom. A cubic polynomial has 10 components. The term $x^2y + xy^2$ is combined to reduce the number of degrees of freedom to nine.

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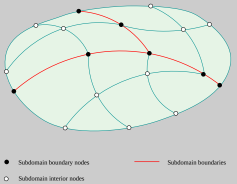

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6.4.2. Summary

Conforming C^1 elements are costly. For rectangular element we have not discussed isoparametric elements, this is not simple.

Nonconforming elements is an alternative. We have seen some, they may also be derived using *Hybride methods*. The *Patch test* should be satisfied.

6.5. Thick plates: Displacement methods

Finite elements for plates are straightforward using elements satisfying C^0 continuity.

The displacements are interpolated using

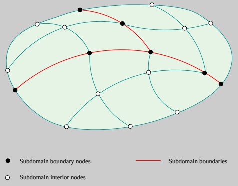
$$w = \boldsymbol{N}_w \boldsymbol{d}_w, \qquad \theta_x = \boldsymbol{N}_\theta \boldsymbol{d}_{\theta_x} \quad \text{and} \quad \theta_y = \boldsymbol{N}_\theta \boldsymbol{d}_{\theta_y}$$

The elemental degrees of freedom are:

$$\boldsymbol{d} = \begin{pmatrix} \boldsymbol{d}_w \\ \boldsymbol{d}_{\theta_x} \\ \boldsymbol{d}_{\theta_y} \end{pmatrix}$$

As we have noted before the stiffness matrix is split into a bending term and a shear terms: tvers:

$$\boldsymbol{k} = \boldsymbol{k}_b(t^3) + \boldsymbol{k}_s(t)$$



where t is the thickness of the plate. When $\boldsymbol{N}_w = \boldsymbol{N}_\theta$ and $h \to 0$ the shear term will dominate. According to thin plate theory, the bending should dominate.

This effect can be reduced by using reduced or selective reduced integration.

It can also be reduced or eliminated using different interpolation functions for the vertical displacement, w, and the rotations, θ_x, θ_y. The *Heterosis* element is an example. Here the eight node *Serendipity* element is used to approximate w, while *Lagrange* interpolation is used for the rotations, θ_x, θ_y. (Higher order elements are also possible, see [Hughes, 1987]).

Different choices of elements are shown in Tabel 6.1.

Element evaluation: ...

1. 9-node Lagrange element can represent pure bending (for all integration rules). Since w have quadratic variation, zero shear deformation ($\gamma_{zx} = 0$) result in pure bending rather than zero bending.

   For linear bending, $w = 0$ and not cubic as required.

2. The Serendipity element is dubious for all integration rules, it does not satisfy the Patch test.

3. The Heterosis element is frequently the best element, combined with selective reduced integration (3×3 for $\boldsymbol{k}_b$ and 2×2 for $\boldsymbol{k}_s$) the results are good and robust.

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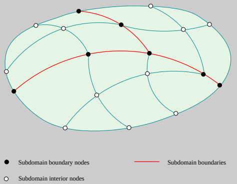


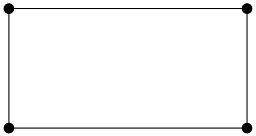
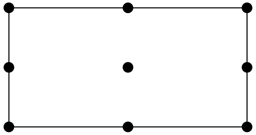
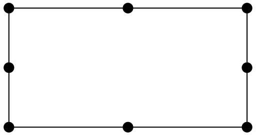
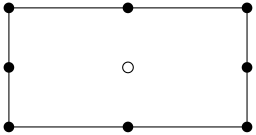
			Integration rule				
Elementtype		Type	$\boldsymbol{k}_b$	$\boldsymbol{k}_s$	n_{sc}	n_{dof}	n_m
	Bilinear	R	1×1	1×1	2	3	4
	4 node	S	2×2	1×1	2	3	2
	12 dof	F	2×2	2×2	8	3	0
	Lagrange	R	2×2	2×2	8	12	4
	9 nodes	S	3×3	2×2	8	12	1
	27 dof	F	3×3	3×3	18	12	0
	Serendipity	R	2×2	2×2	8	9	1
	8 nodes	S	3×3	2×2	8	9	0
	24 dof	F	3×3	3×3	18	9	0
	Heterosis	S	3×3	2×2	8	11	0
	9 nodes						
	26 dof						

Table 6.1: Integration of Mindlin-Reissner plates. Integration: R = reduced, S = selective reduced, F = full integration. n_{sc} = number of shear conditions in an element, n_{dof} = number of degrees of freedom added to a large mesh from one element, n_m = number of conditions in one element.

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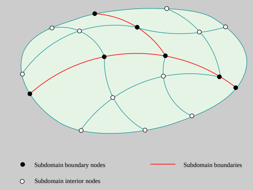

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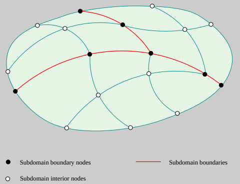


6.6. Discrete Kirchhoff (DK) elements

In this formulation on begins with independent fields for displacements, w, and rotations, (θ_x, θ_y). In this respect it is similar to the Mindlin-Reissner formulation.

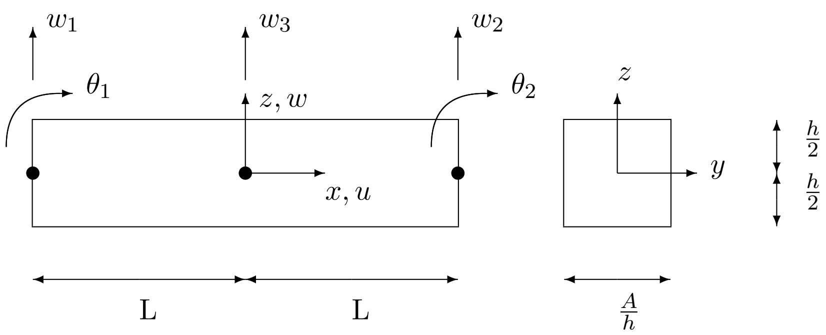
In a DK formulation the shear strains, $\gamma_{xz} = \gamma_{yz} = 0$, are set to zero at certain points in the element. For further details see [Cook et al., 2002][1], C15.2.

At the Department of Mechanics, University of Oslo, plate elements in practical analysis is studied in Bjørge and Mood. The triangular DK element (DKT) are reported to achieve good results.

[1] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition, October 2002.

Øving 6.1

Figuren viser et C^0-type bjelkeelement med rektangulært tverrsnitt.



Virkningen av skjærtøyninger $2\varepsilon_{xz}$ skal tas med under forutsetningen om at plane tverrsnitt forblir plane, men ikke nødvendigvis normale til nøytralaksen etter deformasjon. Videre antas det at spenningene $\sigma_{yy} = \sigma_{zz} = 0$, og at tverrkontraksjonskoeffisienten $\nu = 0$. Forskyvningene i et punkt i avstand z fra nøytralaksen kan da uttrykkes på følgende måte

$$u = z\theta(x) \qquad \text{og} \qquad w = w(x)$$

hvor θ betegner rotasjonen av et tverrsnitt. Det forutsettes at materialet har elastisitetsmod-

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ulen E og skjærmodulen G.

a) Elementet formuleres på basis av

$$w = \boldsymbol{N}_w \boldsymbol{v}_w \qquad \text{der} \qquad \boldsymbol{v}_w = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

og

$$\theta = \boldsymbol{N}_\theta \boldsymbol{v}_\theta \qquad \text{der} \qquad \boldsymbol{v}_\theta = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

Angi interpolasjonsfunksjonene i $\boldsymbol{N}_w$ og $\boldsymbol{N}_\theta$ som funksjoner av den dimensjonsløse koordinaten $\xi = \frac{x}{L}$.

b) Finn $\boldsymbol{B}$-matrisene $\boldsymbol{B}_b$ og $\boldsymbol{B}_s$ som angir bøyetøyningene

$$\varepsilon_b = \varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\boldsymbol{B}_b \boldsymbol{v}$$

i lengderetningen og skjærtøyningen

$$2\varepsilon_s = 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \boldsymbol{B}_s \boldsymbol{v}$$

på tvers av lengdeaksen. $\boldsymbol{v}^T = \{\boldsymbol{v}_w^T, \boldsymbol{v}_\theta^T\}$.

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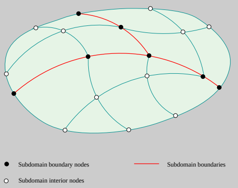

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c) Vis hvordan en ved hjelp av virtuelle forskyvningers prinsipp eller prinsippet om minimum potensiell energi kan komme fram til uttrykket for bøyestivheten $\boldsymbol{k}_b$ og skjærstivheten $\boldsymbol{k}_s$ for elementet. Det er forutsatt at deformasjonen i og på tvers av lengdeakseretingen er ukoblet.

d) Vis hvordan en ved statisk kondensering kan eliminere frihetsgraden w_3. $\boldsymbol{k}_b$ er gitt ved

$$\boldsymbol{k}_b = \frac{EI}{2L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

og $\boldsymbol{k}_s$ er gitt ved

$$\boldsymbol{k}_w = \frac{G\bar{A}}{6L}\begin{bmatrix} 7 & 1 & -8 & -5L & -L \\ 1 & 7 & -8 & L & -5L \\ -8 & -8 & 16 & 4L & -4L \\ -5L & L & 4L & 4L^2 & 2L^2 \\ -L & -5L & -4L & 2L^2 & 4L^2 \end{bmatrix}$$

e) Hvordan kan en forbedre elementets egenskaper når høyden av bjelken $h \to 0$. Hvilke følger kan de foreslåtte forbedringene få for elementstivhetsmatrisen?

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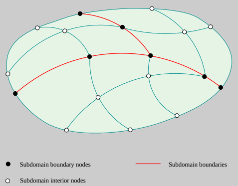

f) Sett opp kravene for at et C^0-bjelkeelement skal tilfredstille fullstendighetskriteriet. Tilfredstiller dette elmentet, slik det er formulert i a), fullstendighetskriteriet? Vil elimineringen av frihetsgraden w_3 ved statisk kondensering endre elementets muligheter for å tilfredsstille fullstendighetskriteriet?

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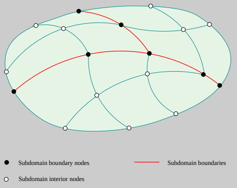

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[Hughes, 1987] Hughes, T. J. R. (1987). *The Finite Element Method, Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey.

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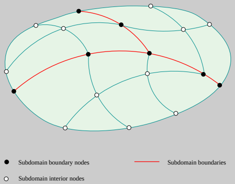

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