24

b

$$
\begin{aligned}
f(x;\theta) &= (\theta+1)x^{\theta} I_{0,1}(x) \\
f(\theta;\vec{x}) &= \prod_i (\theta+1)x_i^{\theta} I_{0,1}(x_i) \\
f(\theta;\vec{x}) &= (\theta+1)^n (\prod_i x_i)^{\theta} I_{>0}(\min_i x) I_{<1}(\max_i(x)) \\
\log(f(\theta;\vec{x})) &= n\log(\theta+1) + \theta \sum_i log(x_i) \\
\frac{\partial \log(f(\theta;\vec{x}))}{\partial\theta} &= n1/(\theta+1) + \sum_i log(x_i) \\
0 &= n/(\hat{\theta}+1) + \sum_i log(x_i) \\
(\hat{\theta}+1)\sum_i log(x_i) &= -n \\
\hat{\theta} &= -n/\sum_i log(x_i) - 1 \\
&= -10/-2.429503 - 1 \\
&= 3.110
\end{aligned}
$$

29

a

$$
\begin{aligned}
f(x;\theta) &= \frac{x}{\theta} e^{-x^2/(2\theta)} \\
f(\theta;\vec{x}) &= \prod_i \frac{x_i}{\theta} e^{-x_i^2/(2\theta)} \\
\log(f(\theta;\vec{x})) &= \sum_i \log(x_i) - n\log\theta - \sum x_i^2/(2\theta) \\
\frac{\partial \log(f(\theta;\vec{x}))}{\partial\theta} &= -n/\theta + \sum x_i^2/(2\theta^2) \\
0 &= -n/\hat{\theta} + \sum x_i^2/(2\hat{\theta}^2) \\
n\hat{\theta} &= \sum x_i^2/2 \\
\hat{\theta} &= \sum x_i^2/2n \\
\hat{\theta} &= 1490.106/20 = 74.51
\end{aligned}
$$

b

Median $medx$ is such that $\int_0^{medx} f(x;\theta)dx = 0.5$ so :

$$\begin{aligned}
\int_0^{m} edx \frac{x}{\theta} e^{-x^2/(2\theta)} dx &= 0.5 \\
\left[-e^{-x^2/(2\theta)}\right]_0^{m} edx &= 0.5 \\
-e^{-medx^2/(2\theta)} + 1 &= 0.5 \\
e^{-medx^2/(2\theta)} &= 0.5 \\
-medx^2/(2\theta) &= \log(0.5) \\
medx &= \sqrt{-\log(0.5)(2\theta)} \\
&= \sqrt{2\log(2)\theta} \\
m\hat{e}dx &= \sqrt{2\log(2)\hat{\theta}} \\
m\hat{e}dx &= \sqrt{2\log(2)74.51} = 10.163
\end{aligned}$$

42

a

$$I(\lambda) = V\left[\frac{\partial \log(f(X;\lambda))}{\partial\lambda}\right]$$

$$\begin{aligned}
f(X;\lambda) &= \lambda^X e^{-\lambda}/k! \\
\log(f(X;\lambda)) &= X\log(\lambda) - \lambda - \sum_{i=1}^{k} \log i \\
\frac{\partial \log(f(X;\lambda))}{\partial\lambda} &= X/\lambda - 1 \\
V\left[\frac{\partial \log(f(X;\lambda))}{\partial\lambda}\right] &= V(X/\lambda - 1) \\
&= 1/\lambda^2 V(X) \\
&= 1/\lambda^2 \lambda = \lambda^{-1}
\end{aligned}$$

b

$$\begin{aligned}
V(T) \geq 1/(nI(\lambda)) &= 1/(n\lambda^{-1}) \\
&= \lambda/n
\end{aligned}$$

c

$$\frac{\partial \log(f(\vec{X};\lambda))}{\partial\lambda} = \sum_i X_i/\lambda - n$$
$$0 = \sum_i X_i/\hat{\lambda} - n$$
$$\hat{\lambda} = \sum_i X_i/n = \bar{X}$$

d

$$E(\hat{\lambda}) = E(\bar{X}) = \lambda$$
$$V(\hat{\lambda}) = V(\bar{X})$$
$$= \lambda/n = 1/(nI(\lambda))$$

43

a

We have that $\hat{\theta} = \max_i X_i = Y_n$ has pdf:

$$g_n(y) = n[F(y)]^{n-1}f(y)$$
$$g_n(y) = n[\int_0^y 1/\theta dt]^{n-1}1/\theta$$
$$g_n(y) = n[y/\theta]^{n-1}1/\theta$$
$$g_n(y) = n[y/\theta]^{n-1}\frac{\partial y/\theta}{\partial y}$$

so $\hat{\theta}/\theta \sim B(n,1)$ and thus $E(\hat{\theta}) = \theta E(\hat{\theta}/\theta) = \theta n/(n+1)$ and $V(\hat{\theta}) = \theta^2 V(\hat{\theta}/\theta) = \theta^2 n/((n+1)^2(n+2))$

$$\tilde{\theta} = (n+1)/n\hat{\theta}$$
$$E(\tilde{\theta}) = (n+1)/nE(\hat{\theta}))$$
$$= (n+1)/n(n/(n+1)\theta) = \theta$$

$$V(\tilde{\theta}) = (n+1)^2/n^2V(\hat{\theta})$$
$$= (n+1)^2/n^2\theta^2 n/((n+1)^2(n+2))$$
$$= \theta^2/(n(n+2))$$

b

$$\begin{aligned}
I(\theta) &= \frac{\partial^2 \log(f(X;\theta))}{\partial\theta^2} \\
\log(f(X;\theta)) &= \log(1/\theta) \\
\log(f(X;\theta)) &= -\log(\theta)) \\
\frac{\partial \log(f(X;\theta))}{\partial\theta} &= -1/\theta \\
\frac{\partial^2 \log(f(X;\theta))}{\partial\theta^2} &= 1/\theta^2 \\
V\left[\frac{\partial \log(f(X;\theta))}{\partial\theta}\right] &= 0 \\
V(T) < 1/(nI(\theta)) &= \theta^2/n
\end{aligned}$$

c

Since θ is a factor in the bounds of the distribution, Cramer-Rao does not hold.

44

a

$$\begin{aligned}
f(x;\mu) &= \frac{1}{\mu}e^{-x/\mu} \\
\log(f(x;\mu)) &= -\log(\mu) - x/\mu \\
\frac{\partial \log(f(x;\mu))}{\partial\mu} &= -1/\mu + x/\mu^2 \\
V\left[\frac{\partial \log(f(x;\mu))}{\partial\mu}\right] &= V(x)/\mu^4 \\
&= \mu^2/\mu^4 = \mu^-2 \\
V(T) &\geq 1/(nI(\mu)) = \mu^2/n
\end{aligned}$$

b

$$\begin{aligned}
S(\mu) &= \frac{\partial \log(f(\vec{X};\mu))}{\partial\mu} \\
&= -n/\mu + \sum_i X_i/\mu^2 \\
S(\hat{\mu}) &= 0 \\
0 &= -n/\hat{\mu} + \sum_i X_i/\hat{\mu}^2 \\
\hat{\mu} &= \sum_i X_i/n = \bar{X}
\end{aligned}$$

c

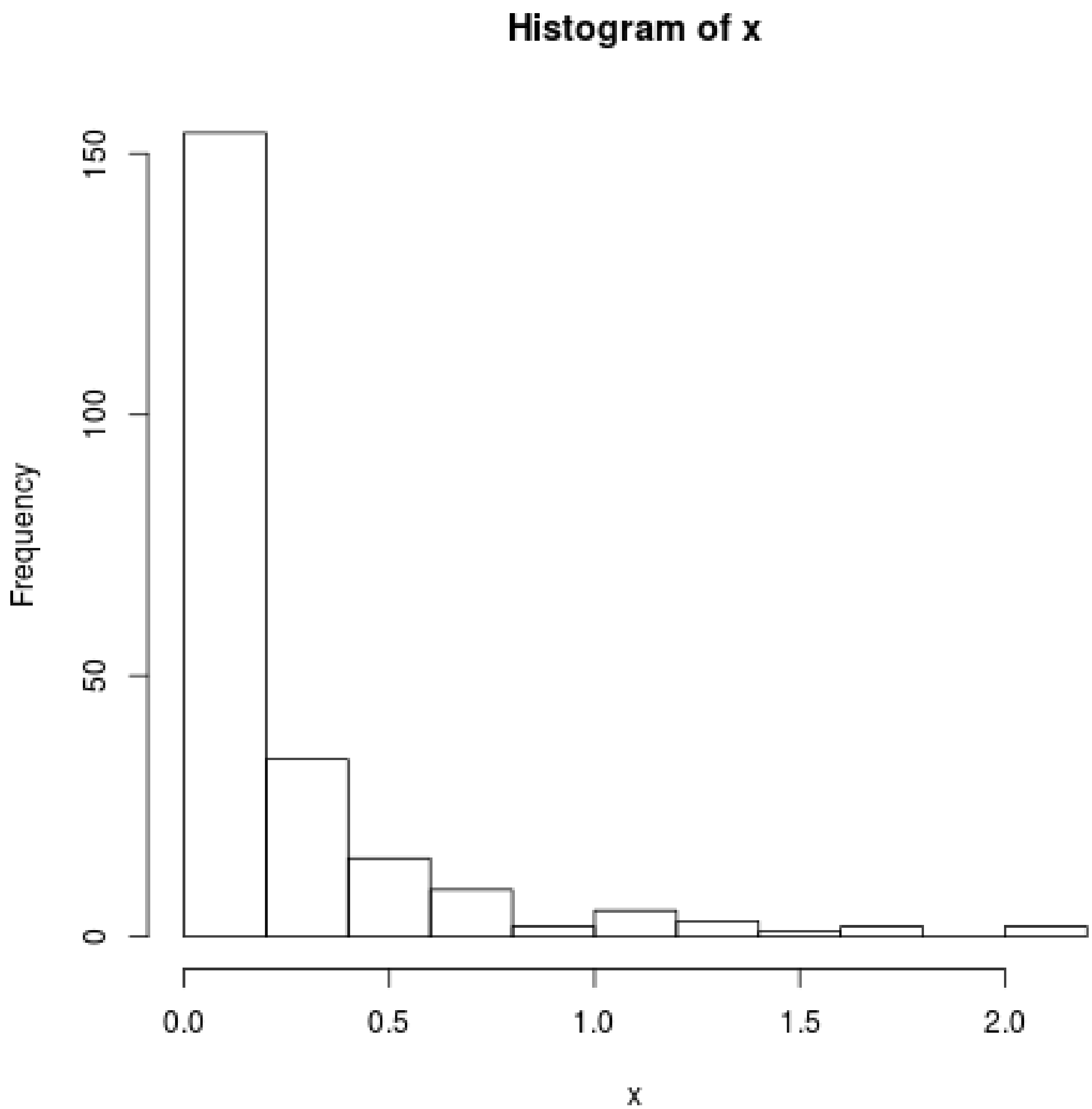
$$
\begin{aligned}
E(\hat{\mu}) &= E(\bar{X}) = \mu \\
V(\hat{\mu}) &= V(\bar{X}) = V(X)/n = \mu^2/n
\end{aligned}
$$

d

It is an efficient estimator because it achieves the C-R lower bound

Ekstraoppgave 7

a



b

$$
\begin{aligned}
f(x \mid \alpha,\lambda) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \\
f(\alpha,\lambda \mid \vec{x}) &= f(\vec{x} \mid \alpha,\lambda) \\
&= \prod_i \frac{\lambda^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\lambda x_i} \\
&= \frac{\lambda^{n\alpha}}{\Gamma(\alpha)^n} \left[\prod_i x_i\right]^{\alpha-1} e^{-\lambda \sum_i x_i} \\
\log(f(\alpha,\lambda \mid \vec{x})) &= n\alpha\log\lambda - n\log\Gamma(\alpha) + (\alpha-1)\sum \log x_i - \lambda \sum_i x_i
\end{aligned}
$$

c

$$
\begin{aligned}
\log(f(\alpha,\lambda \mid \vec{x})) &= n\alpha\log\lambda - n\log\Gamma(\alpha) + (\alpha-1)\sum \log x_i - \lambda \sum_i x_i \\
\frac{\partial \log(f(\alpha,\lambda \mid \vec{x}))}{\partial\alpha} &= n\log\lambda - n/\Gamma(\alpha)\Gamma'(\alpha) + \sum \log x_i \\
\frac{\partial \log(f(\alpha,\lambda \mid \vec{x}))}{\partial\lambda} &= n\alpha/\lambda - \sum_i x_i
\end{aligned}
$$

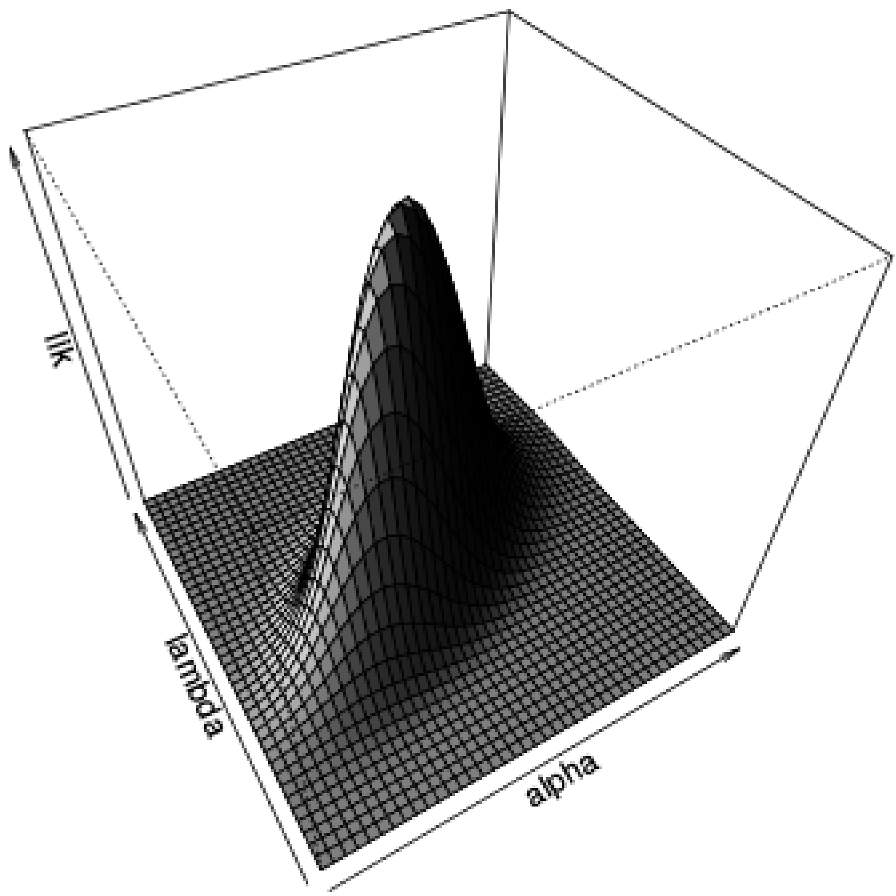
d

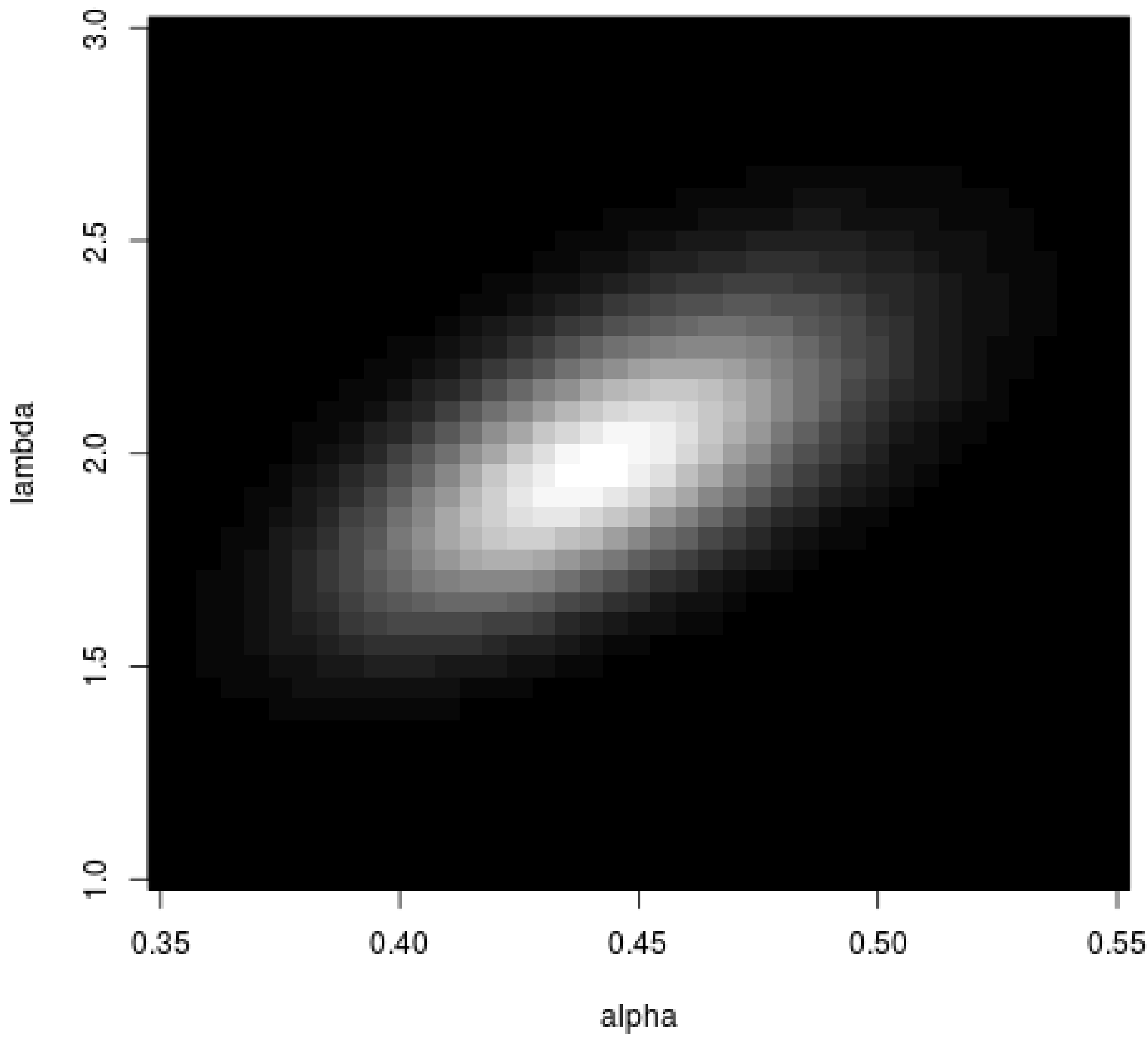
```
x=scan(
    "http://www.uio.no/studier/emner/matnat/math/STK2120/v13/illrain.dat",
    na.strings="*")
x=x[!is.na(x)]
png("ex7hist.png")
hist(x)
dev.off()


alpha = seq(0.35,0.55,0.005)
lambda = seq(1,3,0.05)
loglik = matrix(nrow=length(alpha),ncol=length(lambda))
n = length(x)
sumx=sum(x)
sumlogx = sum(log(x));
for(i in 1:length(alpha))
for(j in 1:length(lambda))
loglik[i,j] = n*alpha[i]*log(lambda[j])+(alpha[i]-1)*sumlogx-lambda[j]*sumx-n*log(gamma(alp
lik=exp(loglik)
png("extra7persp.png")
persp(alpha,lambda,lik,theta=330,phi=45,shade=1,zlab="lik")
dev.off()

png("extra7image.png")
image(alpha,lambda,lik,col=gray((0:32)/32))
dev.off()
```

e





From plot: $\hat{\lambda} \approx 2.0$ and $\hat{\alpha} \approx 0.44$