46

a

$$
\begin{aligned}
f(x;\sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \\
\log(f(x;\sigma^2)) &= -1/2(\log(2)+\log(\pi)+\log(\sigma^2)) - (x-\mu)^2/(2\sigma^2) \\
\frac{\partial \log(f(x;\sigma^2))}{\partial \sigma^2} &= -\frac{1}{2\sigma^2} + (x-\mu)^2/(2\sigma^4) \\
V\left(\frac{\partial \log(f(x;\sigma^2))}{\partial \sigma^2}\right) &= \frac{1}{4\sigma^4} V((x-\mu)^2/\sigma^2) \\
&= \frac{1}{4\sigma^4} 2 \\
&= \frac{1}{2\sigma^4}
\end{aligned}
$$

Cramer Rao :

$$
\frac{1}{nI(\sigma^2)} = \frac{2\sigma^4}{n}
$$

b

$$
\begin{aligned}
s(\sigma^2) &= \sum_i \left[-\frac{1}{2\sigma^2} + (x_i-\mu)^2/(2\sigma^4)\right] \\
s(\sigma^2) &= -\frac{n}{2\sigma^2} + \sum_i (x_i-\mu)^2/(2\sigma^4) \\
s(\hat{\sigma^2}) &= 0 \\
0 &= -\frac{n}{2\hat{\sigma^2}} + \sum_i (x_i-\mu)^2/(2\hat{\sigma^2}^2) \\
n\hat{\sigma^2} &= \sum_i (x_i-\mu)^2 \\
\hat{\sigma^2} &= \frac{\sum_i (x_i-\mu)^2}{n}
\end{aligned}
$$

c

$$
\begin{aligned}
[(X_i-\mu)/\sigma]^2 &\sim \chi^2_1 \quad \forall i \\
\sum_i [(X_i-\mu)/\sigma]^2 &\sim \chi^2_n \\
\hat{\sigma^2} = (\sigma^2/n) \sum_i [(X_i-\mu)/\sigma]^2 &\sim (\sigma^2/n)\chi^2_n
\end{aligned}
$$

d

$$
\begin{aligned}
E(\hat{\sigma^2}) &= (\sigma^2/n)n = \sigma^2 \\
V(\hat{\sigma^2}) &= (\sigma^2/n)^2 2n = 2\sigma^4/n
\end{aligned}
$$

So it is efficient (unbiased and achieves CRLB)

e

It is not in conflict since the chi-square converges to a normal distribution for large n.

Exam 2014, 3

a

$$
\begin{aligned}
f(x;\theta) &= \theta x^{\theta-1} \\
\log(f(x;\theta)) &= \log(\theta) + (\theta-1)\log x \\
\frac{\partial \log(f(x;\theta))}{\partial\theta} &= \frac{1}{\theta} + \log x \\
\frac{\partial^2 \log(f(x;\theta))}{\partial\theta^2} &= -\frac{1}{\theta^2} \\
I(\theta) &= -E\left[\frac{\partial^2 \log(f(x;\theta))}{\partial\theta^2}\right] \\
I_n(\theta) &= \frac{n}{\theta^2}
\end{aligned}
$$

$$
\begin{aligned}
s(\theta) &= \sum_i \frac{1}{\theta} + \log x_i \\
s(\hat{\theta}) &= 0 \\
0 &= \frac{n}{\hat{\theta}} + \sum_i \log x_i \\
\hat{\theta} &= -\frac{n}{\sum_i \log x_i}
\end{aligned}
$$

Asymtopic distribution will be normal with mean $E(\hat{\theta}) \to \theta$ and variance achieving the Cramer Rao lower bound: $V(\hat{\theta}) \to 1/I_n(\theta) = \theta^2/n$. So $N(\theta, \theta^2/n)$

b

$$
\begin{aligned}
&\hat{\theta} = -100/(-22.47) = 4.450378 \\
&\hat{\theta} \pm z_{0.005}\frac{1}{\sqrt{I_n(\hat{\theta})}} \\
&\hat{\theta} \pm 2.576\frac{\hat{\theta}}{\sqrt{n}} \\
4.450378 \pm\ &2.576\frac{4.450378}{10} \\
&(3.30, 5.60)
\end{aligned}
$$

Exam 2012, 4

a

$$
\begin{aligned}
f(x \mid x_m, \theta) &= \theta x_m^\theta x^{-\theta-1} \\
\log(f(x \mid x_m, \theta)) &= log(\theta) + \theta \log x_m - (\theta+1)\log x I_{x>x_m}(x, x_m) \\
\frac{\partial \log(f(x \mid x_m, \theta))}{\partial \theta} &= \frac{1}{\theta} + \log x_m - \log x \\
s(\theta) &= \frac{n}{\theta} + n \log x_m - \sum_i \log x_i \\
s(\hat{\theta}) &= 0 \\
0 &= \frac{n}{\hat{\theta}} + n \log x_m - \sum_i \log x_i \\
\hat{\theta} &= \frac{n}{\sum_i \log x_i - n \log x_m}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log(f(x \mid x_m, \theta))}{\partial \theta^2} &= -\frac{1}{\theta^2} \\
-E\left[\frac{\partial^2 \log(f(x \mid x_m, \theta))}{\partial \theta^2}\right] &= \frac{1}{\theta^2} \\
I_n(\theta) &= \frac{n}{\theta^2}
\end{aligned}
$$

$$
\hat{\theta} \sim_{n\to\infty} N(\theta, \frac{\theta^2}{n})
$$

b

$$
\begin{aligned}
&\hat{\theta} \pm z_{0.025} 1/\sqrt{I_n(\hat{\theta})} = \\
&2.1 \pm 1.96\hat{\theta}/\sqrt{n} = \\
&2.1 \pm 1.96 \cdot 2.1/\sqrt{30} = \\
&(1.349, 2.851)
\end{aligned}
$$

Exam 2006,1

a

$$
\begin{aligned}
f(x_i \mid a_1, a_2, \sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i - a_1 x_{i-1} - a_2 x_{i-2})^2/(2\sigma^2)} \\
L(a_1, a_2, \sigma^2) &= \prod_i \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i - a_1 x_{i-1} - a_2 x_{i-2})^2/(2\sigma^2)} \\
\log(L(a_1, a_2, \sigma^2)) &= -n\frac{1}{2}(log(2) + log(\pi) + log(\sigma^2)) - \sum_i (x_i - a_1 x_{i-1} - a_2 x_{i-2})^2/(2\sigma^2) \\
-2\sigma^2(\log(L(a_1, a_2, \sigma^2)) - \frac{1}{\sqrt{2\pi\sigma^2}^n}) &= \sum_i (x_i - a_1 x_{i-1} - a_2 x_{i-2})^2
\end{aligned}
$$

The last LHS is a monotonically decreasing function for L so minimizing it is equivalent to maximizing L.

b

$$s_1(a_1,a_2) = \frac{\partial}{\partial a_1}\sum_i (x_i - a_1x_{i-1} - a_2x_{i-2})^2 = 2\sum_i x_{i-1}(x_i - a_1x_{i-1} - a_2x_{i-2})$$
$$s_2(a_1,a_2) = \frac{\partial}{\partial a_2}\sum_i (x_i - a_1x_{i-1} - a_2x_{i-2})^2 = 2\sum_i x_{i-2}(x_i - a_1x_{i-1} - a_2x_{i-2})$$
$$s_1(\hat{a}_1,\hat{a}_2) = 0$$
$$0 = 2\left[\sum_i x_ix_{i-1} - \hat{a}_1\sum_i x_{i-1}^2 - \hat{a}_2\sum_i x_{i-2}x_{i-1})\right]0$$

$$s_2(\hat{a}_1,\hat{a}_2) = 0$$
$$0 = 2\left[\sum_i x_ix_{i-2} - \hat{a_1}\sum_i x_{i-1}x_{i-2} - \hat{a_2}\sum_i x_{i-2}^2\right]$$

This is a simple set of two linear equations with two unknowns and is solvable.

$$\frac{\partial}{\partial\sigma^2}\log(L(a_1,a_2,\sigma^2)) = -n\frac{1}{2}(1/\sigma^2) + \sum_i (x_i - a_1x_{i-1} - a_2x_{i-2})^2/(2(\sigma^2)^2)$$
$$n\frac{1}{2}(1/\hat{\sigma^2}) = \sum_i (x_i - a_1x_{i-1} - a_2x_{i-2})^2/(2(\hat{\sigma^2})^2)$$
$$\hat{\sigma^2} = \sum_i (x_i - a_1x_{i-1} - a_2x_{i-2})^2/n$$

The confidence intervals are given by:

$$\hat{a}_1 \pm z_{.025}\sqrt{J_{11}^{-1}} =$$
$$0.7710 \pm 1.96\sqrt{0.0199} =$$
$$(0.494508, 1.047492)$$
$$\hat{a}_2 \pm z_{.025}\sqrt{J_{22}^{-1}} =$$
$$-0.2220 \pm 1.96 * \sqrt{0.0204} =$$
$$(-0.501944, 0.05794399)$$

Since $0 \in CI(a_2)$, it is not unreasonable to omit the a_2 parameter.