Extra 8

a

$$
\begin{aligned}
f(x;\lambda) &= \lambda e^{-\lambda x} \\
L(\lambda;\vec{(x)}) &= \lambda^n e^{-\lambda \sum_i x_i} \\
\log(L(\lambda;\vec{(x)})) &= n\log(\lambda) - \lambda \sum_i x_i \\
\frac{\partial}{\partial \lambda}\log(L(\lambda;\vec{(x)})) &= n/\lambda - \sum_i x_i \\
\hat{\lambda} &= n/\sum_i x_i
\end{aligned}
$$

$$
\begin{aligned}
L(\hat{\Omega}_0) &= \lambda_0^n e^{-\lambda_0 \sum_i x_i} \\
&= \lambda_0^n e^{-\lambda_0 n\bar{x}} \\
L(\hat{\Omega}) &= (n/\sum_i x_i)^n e^{-(n/\sum_i x_i)\sum_i x_i} \\
&= \bar{x}^{-n} e^{-n} \\
\frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} &= \frac{\lambda_0^n e^{-\lambda_0 n\bar{x}}}{\bar{x}^{-n}e^{-n}} = (\lambda_0\bar{x})^n e^{n-\lambda_0 n\bar{x}}
\end{aligned}
$$

b

$$
\begin{aligned}
\log(\frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}) &= n\log(\bar{x}\lambda_0) - n(\lambda_0\bar{x}-1) \\
-2\log(\frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}) &= -2n\log(\bar{x}\lambda_0) + 2n(\lambda_0\bar{x}-1)
\end{aligned}
$$

λ is fixed under H_0, else varying, so we have 1 df.

```
> qchisq(0.95,1)
[1] 3.841459
```

Extra 9

a

$$
\begin{aligned}
f(x;\alpha,\beta) &= \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}e^{-(x/\alpha)^\beta} \\
L(\alpha,\beta) &= \left(\frac{\beta}{\alpha}\right)^n \left(\frac{\prod_i x_i}{\alpha^n}\right)^{\beta-1} e^{-\sum_i (x_i/\alpha)^\beta} \\
l(\alpha,\beta) &= n(\log(\beta)-\log(\alpha)) + (\beta-1)(\sum_i \log(x_i) - n\log(\alpha)) - \sum_i (x_i/\alpha)^\beta \\
&= n\log(\beta) - n\beta\log(\alpha)) + (\beta-1)\sum_i \log(x_i) - \sum_i (x_i/\alpha)^\beta
\end{aligned}
$$

b

$$
\begin{aligned}
s_1(\alpha,\beta) &= \frac{\partial l(\alpha,\beta)}{\partial\alpha} \\
&= -n\beta/\alpha + \beta\sum_i x_i^\beta/\alpha^{\beta+1} \\
s_2(\alpha,\beta) &= \frac{\partial l(\alpha,\beta)}{\partial\beta} \\
&= n/\beta - n\log\alpha + \sum_i \log x_i - \sum (x_i/\alpha)^\beta \log(x_i/\alpha)
\end{aligned}
$$

c

$$
\begin{aligned}
J_{11} &= -\frac{\partial s_1(\alpha,\beta)}{\partial\alpha} \\
&= -n\beta/\alpha^2 + \beta(\beta+1)\sum_i x_i^\beta/\alpha^{\beta+2} \\
J_{12} = J_{21} &= -\frac{\partial s_1(\alpha,\beta)}{\partial\beta} \\
&= n/\alpha - \sum_i x_i^\beta/\alpha^{\beta+1} - \beta/\alpha\sum_i (x_i/\alpha)^\beta \log(x_i/\alpha) \\
J_{22} &= \frac{\partial s_2(\alpha,\beta)}{\partial\beta} \\
&= n/\beta^2 + \sum (x_i/\alpha)^\beta \log(x_i/\alpha)^2
\end{aligned}
$$

d

We assume that the log-likelihood functions looks like a quadratic function around point p and find the extremal point of the quadratic function based on the 1'st and 2'nd derivatives of the log-likelihood function in p. Then repeat the process for the new point p. Repeat

e,f

```
nr.weibull = function(x,theta0,eps=0.000001)
{
    n = length(x);
    sumlogx = sum(log(x));
    diff = 1;theta = theta0;
    alpha = theta[1];beta = theta[2]
    while(diff>eps)
    {
        theta.old = theta
        w1 = sum((x/alpha)^beta)
        w2 = sum((x/alpha)^beta*log(x/alpha))
        w3 = sum((x/alpha)^beta*log(x/alpha)^2)
        s = c(-n*beta/alpha+beta*w1/alpha,
              n/beta-n*log(alpha)+sumlogx-w2)
        Jbar = matrix(c(-n*beta/alpha^2+beta*(beta+1)*w1/alpha^2,
                         n/alpha-w1/alpha-beta*w2/alpha,
```

```
                                    n/alpha-w1/alpha-beta*w2/alpha,n/beta^2+w3),
                                ncol=2)
            theta = theta + solve(Jbar,s)
            alpha = theta[1];beta = theta[2]
            diff = sum(abs(theta-theta.old))
      }
      list(alpha=alpha,beta=beta,Jbar=Jbar)
}


x = c(56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43)

theta = c(16,0.9)

print(nr.weibull(x,theta))

#$alpha
#[1] 17.20194
#
#$beta
#[1] 0.9218849
#
#$Jbar
#                 [,1]         [,2]
#[1,]   0.04595349  -0.4323576
#[2,]  -0.43235761  36.3038771
```

g

$$l(\alpha,\beta) = n\log(\beta) - n\beta\log(\alpha) + (\beta-1)\sum_i \log(x_i) - \sum_i (x_i/\alpha)^\beta$$

$$\begin{aligned} l(\hat\alpha,\hat\beta) &= 16\log(0.9218849) - 16 * 0.9218849\log(17.20194) + (0.9218849-1)36.2322 - 220.3841/17.20194^{0.9218849} \\ &= -62.09617 \end{aligned}$$

h

$$\begin{aligned} s_1(\alpha,\beta) &= -n\beta/\alpha + \beta\sum_i x_i^\beta/\alpha^{\beta+1} \\ s_1(\alpha,1) &= -n/\alpha + \sum_i x_i/\alpha^2 \\ s_1(\hat\alpha_0,1) &= 0 \\ 0 &= -n/\hat\alpha_0 + \sum_i x_i/\hat\alpha_0^2 \\ \hat\alpha_0 n &= \sum_i x_i \\ \hat\alpha_0 &= \bar x \end{aligned}$$

i

$$\hat{\alpha}_0 = \bar{x} = 17.9375 l(\alpha,\beta) \qquad = n\log(\beta) - n\beta\log(\alpha) + (\beta-1)\sum_i \log(x_i) - \sum_i (x_i/\alpha)^\beta$$

$$l(17.9375, 1) = -62.1903$$

j

$$-2\log(\frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} =$$
$$-2l(\hat{\Omega}_0) - 2l(\hat{\Omega}) =$$
$$2*(62.1903 - 62.09617) = 0.18826$$

β is fixed under H_0, but varying in general. α is varying in both casees. So we have 1 df.

```
> pchisq(0.18826,1)
[1] 0.3356312
```

Exam 2008,1

a

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

reject if $|T| > t_{\alpha/2,n-1}$

b

$$f(x \mid \mu,\tau) = \frac{1}{\tau}\frac{e^{(x-\mu)/\tau}}{(1+e^{(x-\mu)/\tau})^2}$$
$$f(\mu+d \mid \mu,\tau) = \frac{1}{\tau}\frac{e^{(\mu+d-\mu)/\tau}}{(1+e^{(\mu+d-\mu)/\tau})^2}$$
$$= \frac{1}{\tau}\frac{e^{d/\tau}}{(1+e^{d/\tau})^2}$$
$$= \frac{1}{\tau}\frac{e^{-d/\tau}}{(e^{-d/\tau}+1)^2}$$
$$= f(\mu-d \mid \mu,\tau)$$

So f is symmetric around $x = \mu$

c

$$
\begin{aligned}
f(x \mid \mu, \tau) &= \frac{1}{\tau}\frac{e^{(x-\mu)/\tau}}{(1+e^{(x-\mu)/\tau})^2} \\
log(f(x \mid \mu, \tau)) &= -\log\tau + (x-\mu)/\tau - 2\log(1+e^{(x-\mu)/\tau}) \\
l(\mu,\tau) &= -n\log\tau + \sum_i (X_i-\mu)/\tau - 2\sum_i \log(1+e^{(X_i-\mu)/\tau}) \\
s_1(\mu,\tau) = \frac{\partial l(\mu,\tau)}{\partial \mu} &= -n/\tau + 2/\tau \sum_i e^{(X_i-\mu)/\tau}/(1+e^{(X_i-\mu)/\tau}) \\
s_2(\mu,\tau) = \frac{\partial l(\mu,\tau)}{\partial \tau} &= -n/\tau - \sum_i (X_i-\mu)/\tau^2 + 2/\tau^2 \sum_i (X_i-\mu)e^{(X_i-\mu)/\tau}/(1+e^{(X_i-\mu)/\tau})
\end{aligned}
$$

d

NR

e

$$\hat{\mu} \sim N(\mu, \bar{J}_{11}^{-1}) = N(\mu, 0.1310)$$

$$Z = \hat{\mu}/\sqrt{J^{-}1_{11}} = 1.0413/\sqrt{0.1310} = 2.877002 > 1.959964 = z_{0.025}$$

So we reject H_0. T-test:

$$T = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} = 1.0495/\sqrt{3.057} * \sqrt{20} = 2.684415 > 2.093024$$

So we reject H_0

f

$$
\begin{gathered}
-2\log(\frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} = \\
-2l(\hat{\Omega}_0) - 2l(\hat{\Omega}) = \\
-2l(0,\tau^*) - 2l(\hat{\mu},\hat{\tau}) \sim \chi_1^2
\end{gathered}
$$

$$
\begin{aligned}
l(\mu, \tau) &= -n \log \tau + \sum_i (X_i - \mu)/\tau - 2 \sum_i \log(1 + e^{(X_i - \mu)/\tau}) \\
&= -n \log \tau + n\bar{X}/\tau - \mu/\tau - 2 \sum_i \log(1 + e^{(X_i - \mu)/\tau}) \\
l(\hat{\mu}, \hat{\tau}) &= -20 \log 0.9453 + 20 * 1.0495/0.9453 - 1.0413/0.9453 - 2 * 20.153 \\
&= -18.07791 \\
l(0, \tau^*) &= -20 \log 1.1425 + 20 * 1.0495/1.1425 - 2 * 29.074 \\
&= -42.44039 \\
-2l(0, \tau^*) - 2l(\hat{\mu}, \hat{\tau}) &= -2 * (-42.44039 + 18.07791) \\
&= 48.72496 > 3.841459
\end{aligned}
$$

So we reject H_0 at $\alpha = 0.05$