91

a

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 4 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$$

b

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \\ 1 \end{pmatrix}$$

c

$$\begin{aligned} \hat{y} &= \mathbf{X}\mathbf{b} \\ &= \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3/2 \\ 1/2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} \\ \mathbf{r} = \hat{y} - y &= \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \\ SSE = \mathbf{r}'\mathbf{r} &= 1 + 1 + 1 + 1 = 4 \\ MSE &= SSE/(n - (k+1) = 4/(4-3) = 4 \end{aligned}$$

d

$$\begin{aligned}&\hat{\beta}_1 \pm t_{.025,1} s\sqrt{c_{11}}\\ =&\hat{\beta}_1 \pm t_{.025,1}\sqrt{MSEc_{11}}\\ =&1/2 \pm 12.706\sqrt{4\cdot 0.25}\\ =&1/2 \pm 12.706\end{aligned}$$

The CI is big since there is only one df

e

$H_0 : \beta_1 = 0$

$$\begin{aligned}T &= \frac{\hat{\beta}_j}{S\sqrt{c_{11}}}\\ &= \frac{1}{2} < t_{.025,1} = 12.706\end{aligned}$$

So we do not reject H_0. Since $0 \in CI$ this is not surprising.

f

Source	df	SS	MS	F
Regression	k	SSR	MSR	f_R
Residual	$n-(k+1)$	SSE	MSE	
Total	$n-1$			

$$\begin{aligned}\bar{y} &= 6/4 = 1.5\\ SST &= (\mathbf{y}-1.5)(\mathbf{y}-1.5)'\\ &= 0.5^2+0.5^2+1.5^2+2.5^2\\ &= 9\\ SSR &= (\hat{\mathbf{y}}-1.5)(\hat{\mathbf{y}}-1.5)'\\ &= 1.5^2+0.5^2+0.5^2+1.5^2\\ &= 5\end{aligned}$$

Source	df	SS	MS	F
Regression	2	5	2.5	0.625
Residual	1	4	4	
Total	3	9		

$f_R = 0.625 < F_{.05,2,1} = 199.5$ So we cannot reject H_0.

$$R^2 = 1 - SSE/SST = 1 - 4/9 = 5/9$$

So five ninths of the variation is explained by the linear regression model.

94

a

$$\mathbf{X} = \begin{pmatrix} 1 & x_1 - \bar{x} \\ 1 & x_2 - \bar{x} \\ \vdots & \vdots \\ 1 & x_n - \bar{x} \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} n & \sum_i (x_i - \bar{x}) \\ \sum_i (x_i - \bar{x}) & \sum_i (x_i - \bar{x})^2 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & S_{XX} \end{pmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 1/n & 0 \\ 0 & 1/S_{XX} \end{pmatrix}$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_i y_i \\ \sum_i (x_i - x) y_i \end{pmatrix} = \begin{pmatrix} \sum_i y_i \\ S_{XY} \end{pmatrix}$$

$$\begin{aligned}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} &= \begin{pmatrix} 1/n & 0 \\ 0 & 1/S_{XX} \end{pmatrix} \begin{pmatrix} \sum_i y_i \\ S_{XY} \end{pmatrix} \\ &= \begin{pmatrix} \bar{y} \\ S_{XY}/S_{XX} \end{pmatrix}\end{aligned}$$

b

$$\hat{\beta}_0 \pm t_{.025, n-(k+1)} s \sqrt{c_{00}} = \bar{y} \pm t_{.025, n-2} s / \sqrt{n}$$
$$\hat{\beta}_1 \pm t_{.025, n-(k+1)} s \sqrt{c_{11}} = S_{XY}/S_{XX} \pm t_{.025, n-2} s / \sqrt{S_{XX}}$$

c

$\mathbf{X}'\mathbf{X}$ is diagonal, so it is easily invertible.

95

a

$$\mathbf{X} = \begin{pmatrix} 1 & 0.5 \\ 1 & 0.5 \\ \vdots & \vdots \\ 1 & 0.5 \\ 1 & -0.5 \\ \vdots & \vdots \\ 1 & -0.5 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 2n & 0 \\ 0 & 2n(0.5)^2 \end{pmatrix} = \begin{pmatrix} 2n & 0 \\ 0 & n/2 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_i y_i \\ 0.5\sum_{i=1}^{n} y_i - 0.5\sum_{i=n}^{2n} y_i \end{pmatrix} = \begin{pmatrix} n(\bar{y}_1 + \bar{y}_2) \\ n(\bar{y}_1 - \bar{y}_2)/2 \end{pmatrix}$$

$$\begin{aligned}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} &= \begin{pmatrix} 1/(2n) & 0 \\ 0 & 2/n \end{pmatrix}\begin{pmatrix} n(\bar{y}_1 + \bar{y}_2) \\ n(\bar{y}_1 - \bar{y}_2)/2 \end{pmatrix} \\ &= \begin{pmatrix} (\bar{y}_1 + \bar{y}_2)/2 \\ \bar{y}_1 - \bar{y}_2 \end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}\hat{\mathbf{y}} &= \mathbf{X}\hat{\beta} \\ &= \begin{pmatrix} 1 & 0.5 \\ 1 & 0.5 \\ \vdots & \vdots \\ 1 & 0.5 \\ 1 & -0.5 \\ \vdots & \vdots \\ 1 & -0.5 \end{pmatrix}\begin{pmatrix} (\bar{y}_1 + \bar{y}_2)/2 \\ \bar{y}_1 - \bar{y}_2 \end{pmatrix} = \begin{pmatrix} (\bar{y}_1 + \bar{y}_2)/2 + (\bar{y}_1 - \bar{y}_2)/2 \\ (\bar{y}_1 + \bar{y}_2)/2 + (\bar{y}_1 - \bar{y}_2)/2 \\ \vdots \\ (\bar{y}_1 + \bar{y}_2)/2 + (\bar{y}_1 - \bar{y}_2)/2 \\ (\bar{y}_1 + \bar{y}_2)/2 - (\bar{y}_1 - \bar{y}_2)/2 \\ \vdots \\ (\bar{y}_1 + \bar{y}_2)/2 - (\bar{y}_1 - \bar{y}_2)/2 \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_1 \\ \vdots \\ \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}SSE &= (\mathbf{y} - \hat{\mathbf{y}})'(\mathbf{y} - \hat{\mathbf{y}}) \\ &= \sum_{i=1}^{m}(y_i - \bar{y}_1)^2 + \sum_{i=m+1}^{m+n}(y_i - \bar{y}_2)^2 \\ &= S_{Y_1Y_1} + S_{Y_2Y_2}\end{aligned}$$

$$\begin{aligned}s = \sqrt{MSE} &= \sqrt{SSE/(2n-2)} \\ &= \sqrt{(S_{Y_1Y_1} + S_{Y_2Y_2})/(2n-2)}\end{aligned}$$

$$c_{11} = 2/n$$

c

$$\hat{\beta}_1 \pm t_{.025,n-(k+1)} s\sqrt(c_{11})$$
$$\bar{y}_1 - \bar{y}_2 \pm t_{.025,n+m-2}\sqrt{(\sum_{i=1}^{m}(y_i-\bar{y}_1)^2 + \sum_{i=m+1}^{m+n}(y_i-\bar{y}_2)^2)/(m+n-2)}\sqrt{2/n}$$

d

$$\bar{y}_1 = (117+119+127)/3 = 121$$
$$\bar{y}_2 = (129+138+139)/3 = 135.34$$
$$\hat{\beta}_0 = (\bar{y}_1 + \bar{y}_2/2) = 128.17$$
$$\hat{\beta}_1 = \bar{y}_1 - \bar{y}_2 = -14.34$$
$$\hat{\mathbf{y}} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_1 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_2 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} 121 \\ 121 \\ 121 \\ 135.34 \\ 135.34 \\ 135.34 \end{pmatrix}$$

$$\begin{aligned} SSE &= \sum_{i=1}^{m}(y_i-\bar{y}_1)^2 + \sum_{i=m+1}^{m+n}(y_i-\bar{y}_2)^2 \\ &= (117-121)^2 + (119-121)^2 + (127-121)^2 + (129-135.34)^2 + (138-135.34)^2(139-135.34)^2 \\ &= (117-121)**2 + (119-121)**2 + (127-121)**2 + (129-135.34)**2 + (138-135.34)**2 + (139-135.34)** \\ &= 116.67 \end{aligned}$$

$$s = \sqrt{SSE/(2n-2)} = 5.40$$
$$c_{11} = 2/n = 2/3$$

CI:

$$\begin{aligned} &\bar{y}_1 - \bar{y}_2 \pm t_{.025,n+m-2}\sqrt{(\sum_{i=1}^{m}(y_i-\bar{y}_1)^2 + \sum_{i=m+1}^{m+n}(y_i-\bar{y}_2)^2)/(2n-2)}\sqrt{2/n} \\ &-14.34 \pm t_{.025,4} 5.40\sqrt{1/3} \\ &-14.34 \pm 12.24 \end{aligned}$$

89

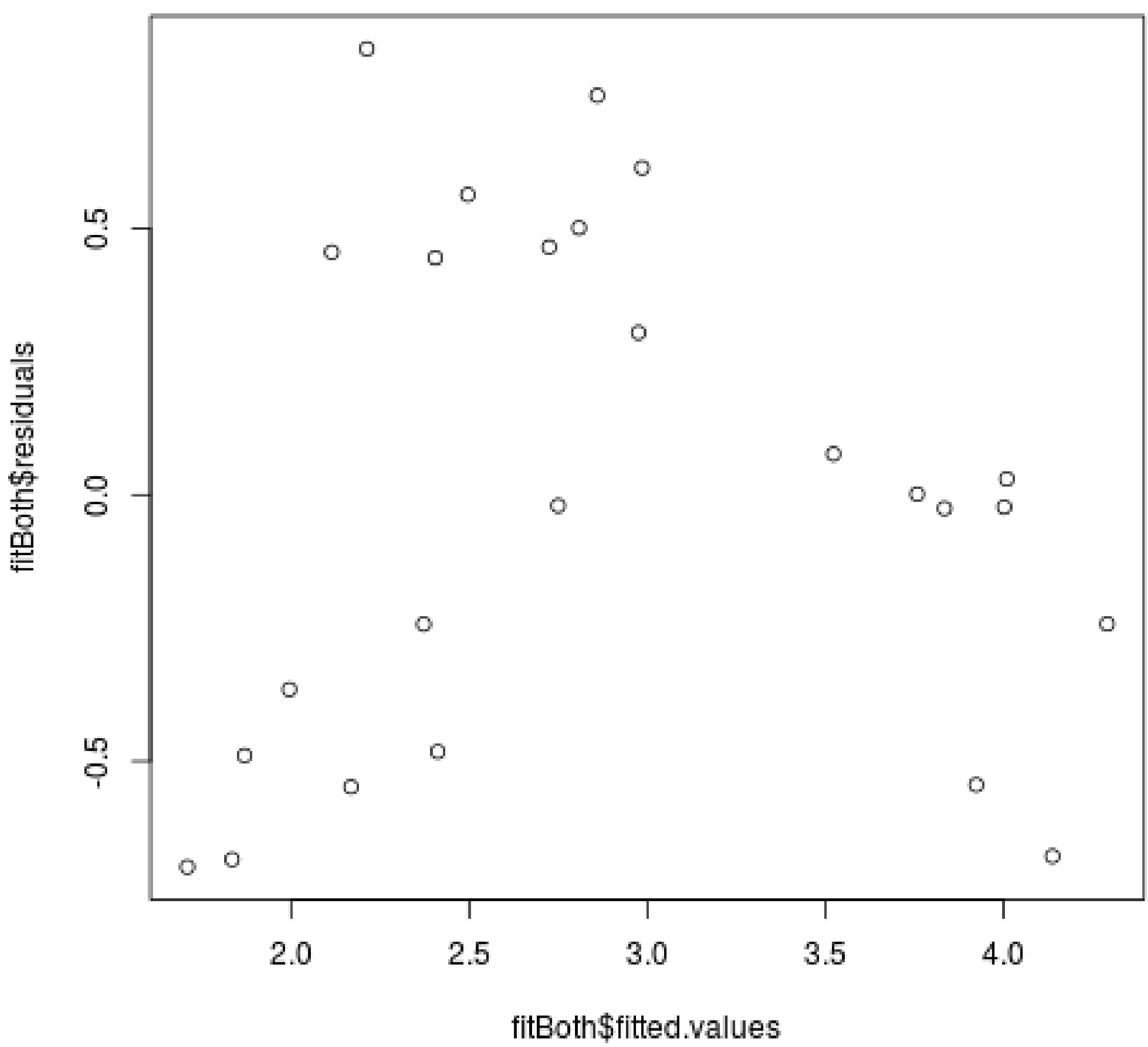
```
exe12.89=read.table(
    "http://www.uio.no/studier/emner/matnat/math/STK2120/v16/exe12.89.txt",
    header=T)
fitIBU = lm(Rating~IBU, data=exe12.89)
summary(fitIBU)

#Coefficients:
```

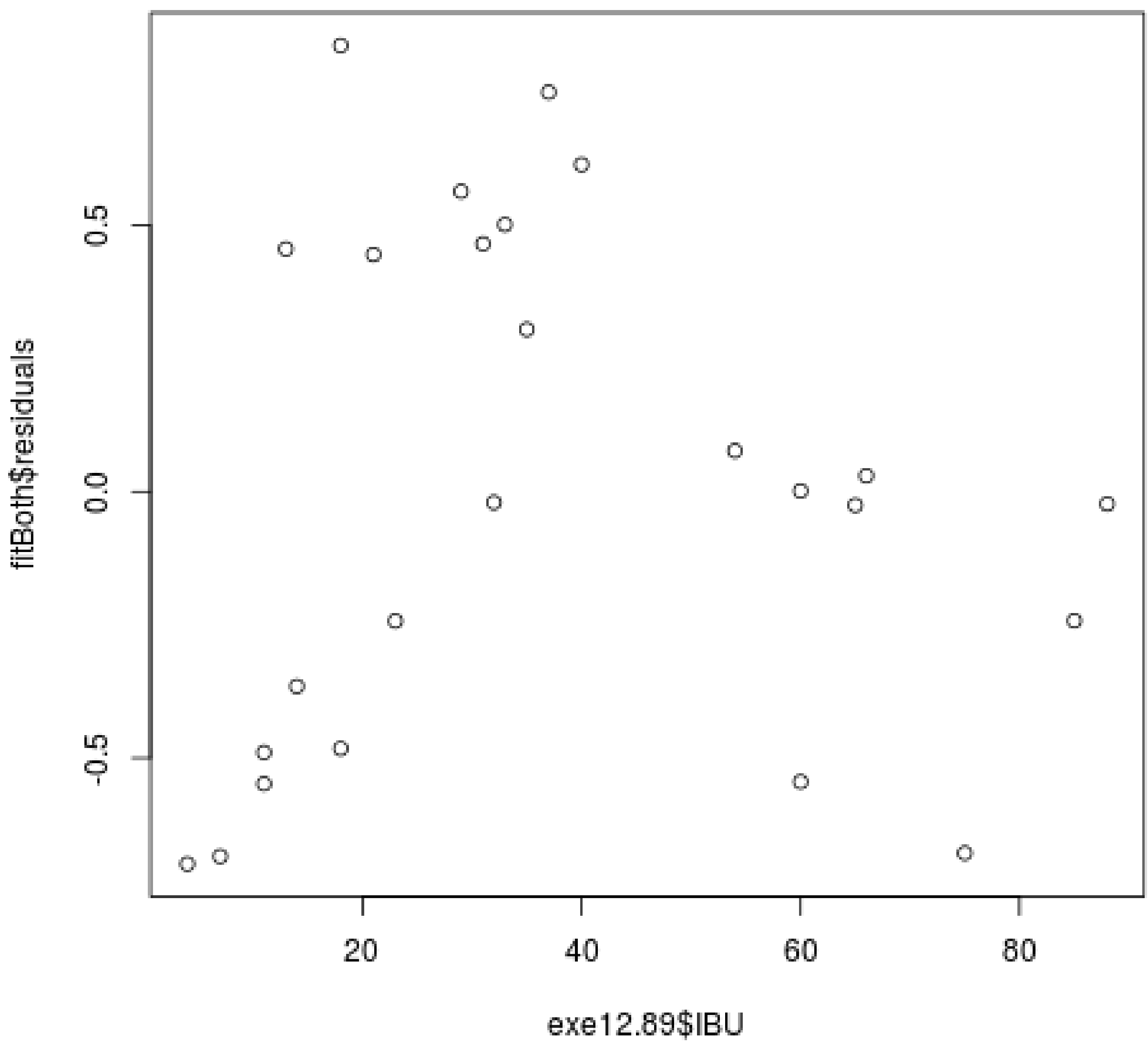
```
#                Estimate Std. Error t value Pr(>|t|)
#(Intercept) 1.698363    0.189611    8.957 5.85e-09 ***
#IBU         0.031958    0.004253    7.514 1.24e-07 ***

fitABV = lm(Rating~ABV, data=exe12.89)
summary(fitABV)

#Coefficients:
#                Estimate Std. Error t value Pr(>|t|)
#(Intercept)  1.07556     0.50090    2.147 0.042544 *
#ABV          0.33011     0.08695    3.797 0.000931 ***
#

                                              #b
fitBoth = lm(Rating~ABV+IBU, data=exe12.89)
summary(fitBoth)
#Coefficients:
#                Estimate Std. Error t value Pr(>|t|)
#(Intercept)  2.238326    0.396128    5.651 1.11e-05 ***
#ABV         -0.166055    0.107849   -1.540    0.138
#IBU          0.041940    0.007688    5.455 1.76e-05 ***
#
# IBU is correleated with ABV and IBU explains rating better

                                              #c
png("exe1289-1.png")
plot(fitBoth$fitted.values,fitBoth$residuals)
dev.off()

png("exe1289-1b.png")
plot(exe12.89$IBU,fitBoth$residuals)
dev.off()

png("exe1289-1c.png")
plot(exe12.89$ABV,fitBoth$residuals)
dev.off()

png("exe1289-2.png")
plot(exe12.89$ABV,exe12.89$Rating)
dev.off()

png("exe1289-3.png")
plot(exe12.89$IBU,exe12.89$Rating)
dev.off() #Not linear

                                              #d
exe12.89$IBUsq = exe12.89$IBU^2
fitQuad = lm(Rating~ABV+IBU+IBUsq, data=exe12.89)
summary(fitQuad)
#Coefficients:
#                 Estimate Std. Error t value Pr(>|t|)
#(Intercept)  0.2142014   0.5181218    0.413 0.683491
#ABV          0.1310974   0.1001233    1.309 0.204558
#IBU          0.0953313   0.0126904    7.512 2.22e-07 ***
```

#IBUsq   −0.0008014   0.0001716   −4.671 0.000131 ***

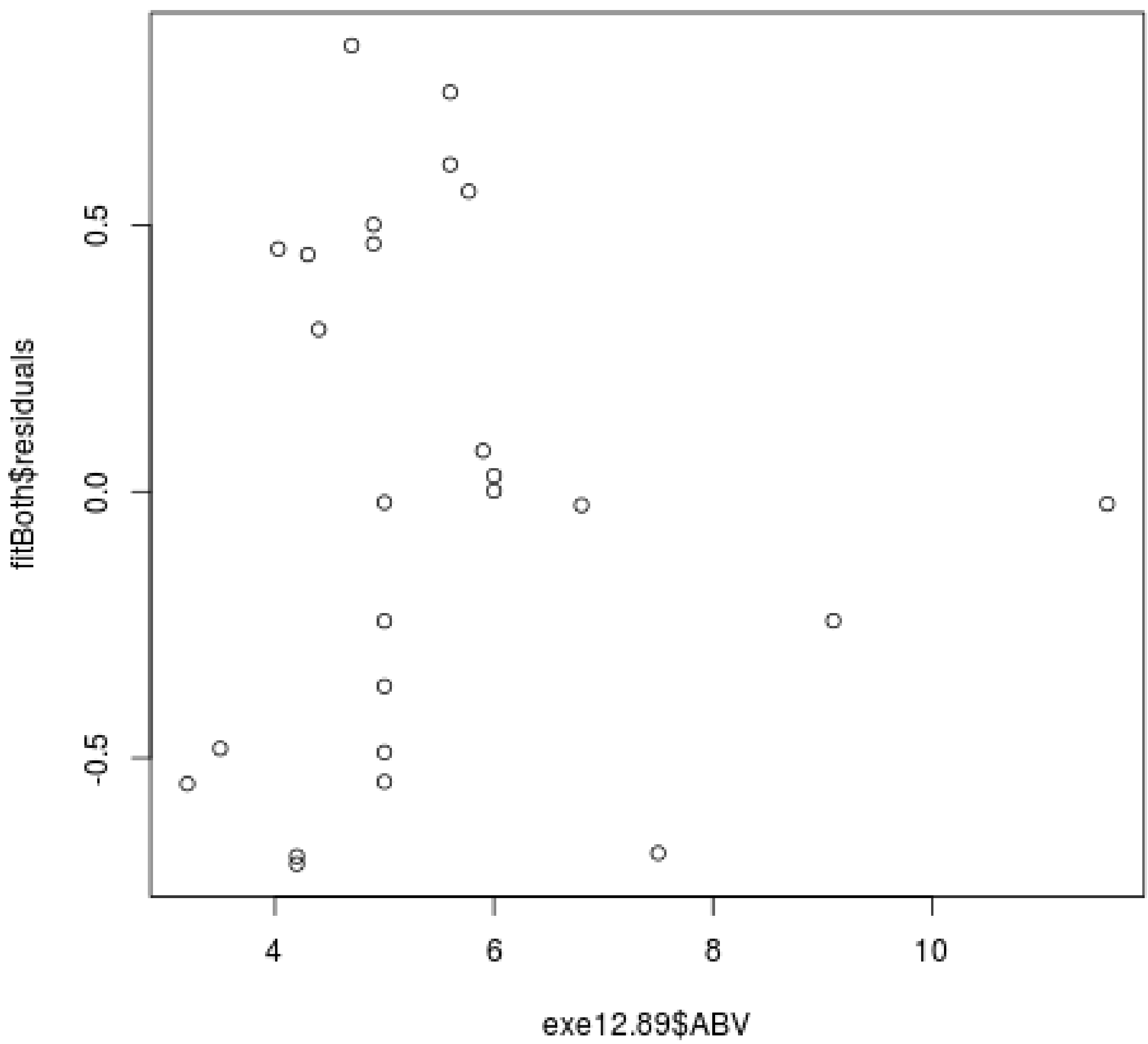
IBU and IBUSq are significant. IBUsq has negative coefficient because of the downward slo

f Rating depends significantly on IBU and IBU^2. When including IBU in the model,
ABV is not significant.
Adjusted R increases from 0.715 to 0.8535 when IBU^2 is included so including
IBU^2 gives a better model

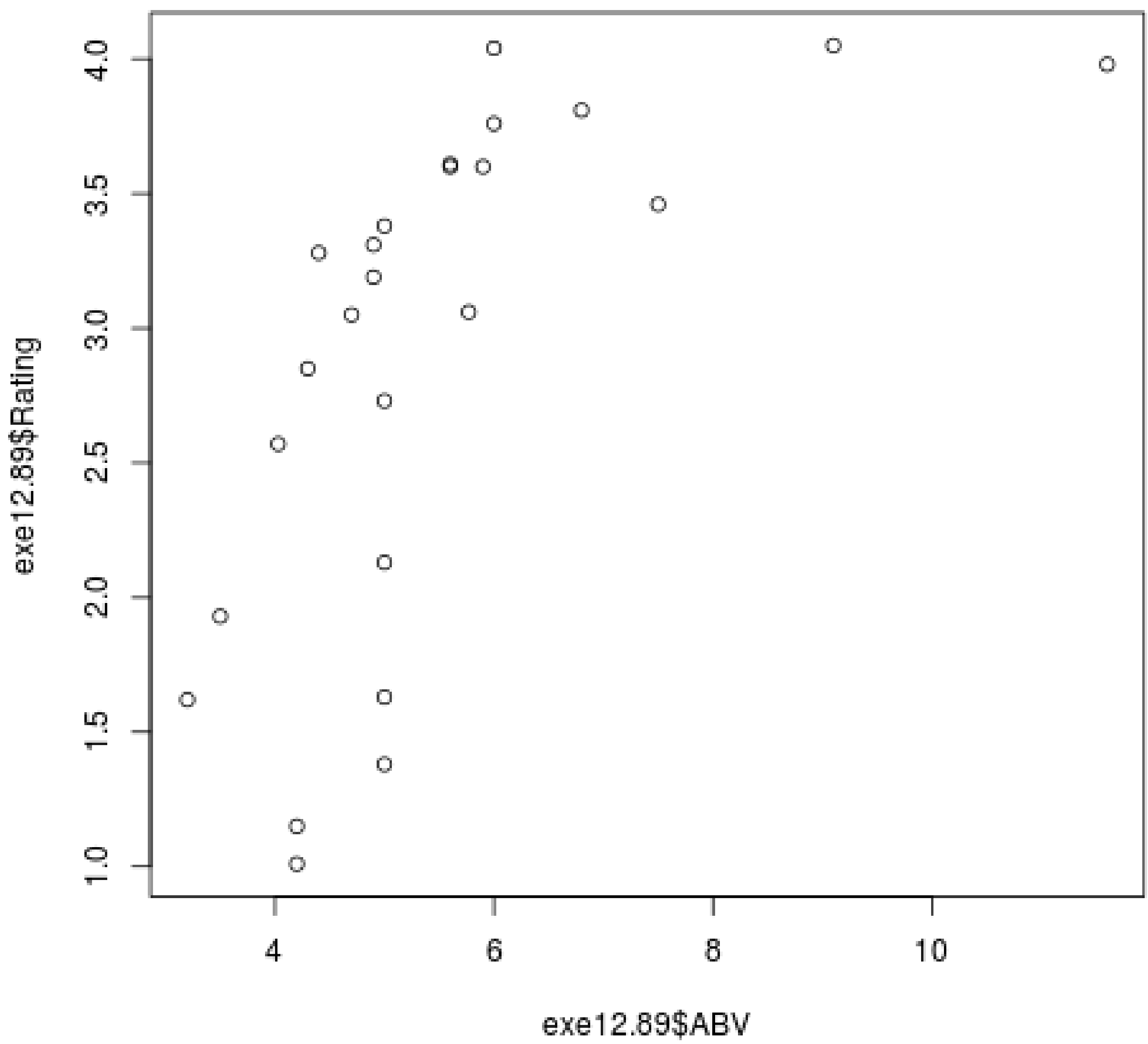




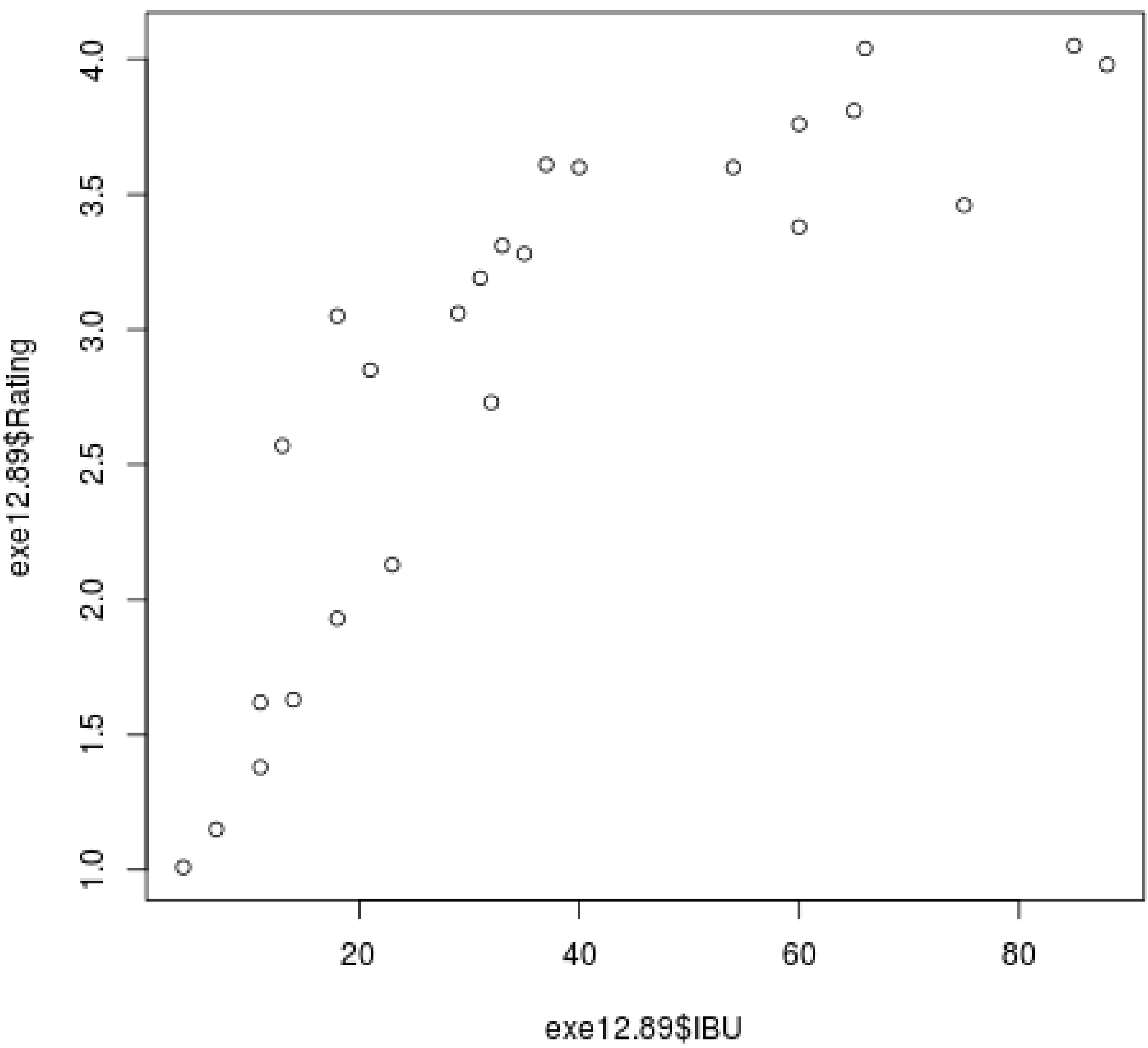
fitBoth$residuals
0.5
0.0
-0.5
20
40
60
80
exe12.89$IBU



0.5
0.0
-0.5
4
6
8
10
fitBoth$residuals
exe12.89$ABV



4.0
3.5
3.0
2.5
2.0
1.5
1.0
exe12.89$Rating
4
6
8
10
exe12.89$ABV



Exam 2

a

(i)

$$r = \sqrt{R^2} = \sqrt{0.4618} = 0.6976$$

(ii)

$$\begin{aligned}
&\hat{\beta}_1 \pm t_{\alpha/2,n-2} s_{\hat{\beta}_1} \\
&0.058514 \pm t_{0.05/2,152} s_{\hat{\beta}_1} \\
&0.058514 \pm 2.8486 \cdot 0.005124 \\
&0.058514 \pm 0.014596 \\
&[0.0439, 0.0731]
\end{aligned}$$

$0.04 \notin$ CI, so we reject H_0

b

The t-test shows that gender is significant. But it is not a significant factor in the regression when we also include height as a factor. Since it is correleated with height, its significance is overrated in the T-test.

c

$$\begin{aligned}
\frac{\partial FEV1}{\partial bmi} &= 0 \\
2(-0.003373)bmi + 0.203648 &= 0 \\
bmi &= \frac{0.203648}{2 \cdot 0.003373} \\
bmi &= 30.1880
\end{aligned}$$

It can be explained by $\frac{\partial FEV1}{\partial bmi} < 0$ for $bmi > 30$ so a positive linear function of bmi will be 'wrong' for $bmi > 30$, which is about a quarter of the persons.

d

(i)

$$\begin{aligned}
&E[\hat{\beta}] \\
=&E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}] \\
=&(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E[\mathbf{Y}] \\
=&(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta \\
=&(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})\beta \\
=&\beta
\end{aligned}$$

(ii)

$$\begin{aligned}
&\Sigma(\hat{\beta}) \\
&\Sigma((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}) \\
&(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Sigma(\mathbf{Y})((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')' \\
&(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\sigma^2\mathbf{I} \\
&(\mathbf{X}'\mathbf{X})^{-1}\sigma^2
\end{aligned}$$