1 Introduction

1.1 A view on the evaluation of risk

The role of mathematics

What skills should the modern actuary possess? Is this influenced by the powerful modern computers? The present book takes the view that an actuary is first and foremost a practitioner who is there to solve practical problems in insurance and finance. Mathematics is an essential part and actually plays two different roles. One is as vendor of *models* which provide simplified descriptions of the complicated world of risk in insurance and finance. These models are usually stochastic. They might in general insurance come as probability distributions for claim frequency and claim size, in life-or disability insurance the central quantities might be Markov processes based on mortalities or intensities of passing from an active state to an disabled one. There are countless other examples.

Mathematics can from this point of view be seen as a language to express statements of risk in, and it is a language the actuaries *must* master. Otherwise they will not be able to to understand how their risk models relate to the reality, what their conclusions mean and neither will they be effective in presenting their analyzes to clients. Actuarial science is in this sense almost untouched by the modern computational facilities. The basic concepts and models remain what they were, notwithstanding, of course, the strong growth of risk products throughout the last decades. This development may have had something to do with computers, but not much with computing per se.

But mathematics is also *deductions*. Through mathematics we derive statements from assumptions, utilizing the rules of logic. At school and at introductory courses at university this is the way mathematics usually is taught. It is here computing enter applied mathematical disciplines like actuarial science. More and more of these deductions are implemented in computers and carried out there. This has been going on for several decades, and it has during that time been an enormous growth in computing power, seemingly with no end. The impact of this technological development is that we may employ simpler and more general computational methods which require less of users.

Risk methodology

The supreme example of such an all-purpose computational technique is *stochastic simulation*, which reproduces in the computer simplified versions of the risk processes that take place in the market place. We shall throughout the book denote risk by letters such as X and Y. They are future payments or payment streams with uncertainty attached. In general insurance they come as compensations for claims or sum of claims. With pensions they are agreed payments interrupted (say) in case of death. Financial risk deals with the future value of assets such as shares and bonds and also with *derived* products where the pay-off are modified from that of the underlying asset through certain contract clauses. Such secondary products exist in insurance too. Re-insurance is the central example.

The mathematical approach, today unanimously accepted, is through probabilities. Risks

The world of risk	In the computer	Sources for $\widehat{M}$
		Historical experience
$M \rightarrow X$	$\widehat{M} \rightarrow X^*$	The implied market view
	$\uparrow$	Deductions from no-arbitrage
	Assumed mechanism	Judgment, physical modelling

Figure 1.1 The working process: Steps when evaluating a risk X.

X and Y are then regarded as *random variables*. Eventually (after the event) we shall *know* what they have become. However, for planning and control and to price risk-taking activities they are needed in advance. We must then fall back on their probability distributions. The working process is depicted in Figure 1.1. On the left is *the real world*, an enormously complicated mechanism (denoted M) that will eventually produce a future X. We shall never know M, though our paradigm is that it *exists* as a mathematical model. What we do instead is to build a *simplified* version $\widehat{M}$ and use it to draw conclusions. These deductions rarely purport to say what X is going to be. The aim may be the expected value (often quoted as the value *today*) or how low X can reasonable fall under unlucky circumstances (which is used for *control*). Even those projections will be wrong if the model $\widehat{M}$ deviates too strongly from M. That issue is a very serious one indeed, and Chapter 6 provides an introduction.

What there is to go on when $\widehat{M}$ is put up is listed on the right in Figure 1.1. Learning from the past is an obvious source (but not all of it is relevant). In finance there is information in the current asset prices which harbour a market view on future development. This so-called *implied view* is discussed in Section 1.4 (and Chapter 12). Then there is the so-called theory of *arbitrage* where the evaluation of derived products in finance partially rests on the assumption that risk-less financial income is impossible. This innocent looking *no-arbitrage* condition has wide implications; see Chapter 13. Personal judgment in parts of all this is unavoidable, but it isn't easy to be specific, and neither shall we go into the physical modelling used (for example) by the big re-insurance firms where simulation models try to imitate actual physical damages in the computer[1]. This book is about how $\widehat{M}$ is constructed from the three other sources, how it is implemented in the computer and how the computer model is used to infer the probability distribution of X.

The computer model

The real risk variable X will only occur once. The economic result of an financial investment in a particular year is an unique event. The same is the aggregated claim against an insurance portfolio during a certain period of time. That is what is different with the computer model. We can play it as many times as we please once we have set it up. Suppose it is used m times to produce m different realizations $X_1^*, \ldots X_m^*$ of X. Clearly we may from this set of *simulated* or *sampled* values read off which values of X are the likely ones and how bad the situation might be if we are unlucky. The *-marking will be used throughout to distin-

[1]Examples are hurricanes ravaging the South-Eastern part of the United States at irregular intervals or earthquakes in vulnerable zones of this planet. Through extensive (and costly) computer simulations of such events, analysts hope to gain insight into their cost.

guish computer simulations from real variables and m will denote the number of simulations.

The method portrayed on the left in Figure 1.1 is known as the *Monte Carlo* method. It will interchangeably be referred to as a method of *sampling* or as *stochastic simulation*. Whatever name used it belongs to the class of techniques for numerical integration, and it has as such a long history; see Evans and Schwarz (2000) for a summary of this important branch of numerical mathematics. Monte Carlo integration is computationally slow, but other numerical methods (that might do the job faster) often require more expertise to use, and they have the weakness that they bog down for high-dimensional integrals, which are precisely what we often encounter when evaluating risk. The Monte Carlo method is unique in tackling well the numerical integration of many variables.

Monte Carlo: General points

What is the significance of numerical speed anyway? Does it really matter that some specialized technique (demanding more time and know-how to implement) is (say) one hundred times faster as long as the one we use still only takes a second? Of course, if the procedure for some reason is to be repeated in a loop thousands of times, it *would* matter. However, the slow simulation techniques are in a huge number of situations quite enough, and, indeed, the practical limit to their use is moving steadily as computers become more and more powerful. How far have we got? The personal computer on the author's desk (not particularly advanced) could produce (2004) around 3 million drawings from the Pareto distribution per second (using Algorithm 2.6, implemented in old-fashioned Fortran) and a similar number from the normal (Algorithm 2.2). That is 1000 claims in an insurance portfolio simulated 10000 times (i.e. 10 *million* draws) completed in about three seconds!

One of the aims of this book is to demonstrate how these opportunities are utilized; i.e. how simulations programs are designed, how they are modified to deal with related (but different) problems and how different programs are merged to handle complex situations where risk is the product of several contributing factors. The versatility and usefulness of the Monte Carlo tool is indicated in Section 1.5 below (and in Chapter 3). By mastering it you may free yourself from what has been preprogrammed in large (and often expensive) software packages. Implementation may not take long if you know the ropes.

What remains is the programming platform to use. This book takes no stand there. All algorithms are written in the pseudo-code of Algorithm 1.1 below. Excell and Visual Basic are a standard in the insurance industry and has become feasible even with simulation. Much higher speed is obtained with C, Pascal or Fortran, and in the opinion of this author people are well adviced to learn simple programming software like those. There are other possibilities, as well. Much can usually be achieved with a platform you know!

1.2 Insurance risk: Basic concepts

Insurance and ceding

General insurance is reimbursements of accidental damages or injuries. Behind this is the notion of **ceding**, i.e. that risk is transferred from one party to another. A typical case is a house owner guarding against destruction by a fire or some natural catastrophe, such as an earthquake or a flood. If an unfortunate event of this kind occurs, compensation is received from a company. The amount X payed during some budget period (typically a year) is either $X = 0$, if no incident is reported or some sum given by the extent of the damage *and* the contract (known as a **policy**). More than one accident is possible.

The ceding of risk is not limited to client and company. It also takes place (and on a routine basis too) *between* companies. This is known as *re-insurance*. The rationale *could* be the same; i.e. a a financially weaker agent transferring risk to a stronger one, but in reality even the largest of companies do this to spread risk. Financially the cedent may be as strong as the other party. Re-insurance is different from ordinary insurance in that the risk originates futher back in a chain of ceding. In mathematical terms we express it through

$$X^{\text{re}} = H(X) \tag{1.1}$$

where H is some function defined by the re-insurance contract; for specific examples, see Section 3.2. Clearly $X^{\text{re}} \leq X$; i.e. the responsibility of the re-insurer is always *less* than the original claim. The rest $X - X^{\text{re}}$ stays with the ceding company.

In practice risk can be ceded several times in complicated networks extended around the globe. This is a tool used by managers to tune their portfolios to a desired risk profile. The contractual obligations define functions such as $H(x)$ in (1.1). Modern actuarial science provides us with means to analyze the risk taken by an agent far removed through intermediaries from the primary source.

Life insurance and pension schemes are handled by the same approach; only the interpretation of X changes. *Term* insurance where a one-time sum is received by the beneficiary of the insured upon the death of the latter, resembles property insurance in that a rare event leads to payment. *Pension schemes* are the opposite. Now compensation may cease if the insured dies. The fact that the likelihood of no payment ($X = 0$) now is small makes no difference for the *method* which remains the same. One of the strong points of mathematics is that many different situations are treated by the same methodology.

The pricing of risk

Of course, transfer of risk, as captured by some random variable X, does not take place for free. The price charged by the insurer or re-insurer (usually in advance) depends, as always, on market conditions, but there is a guideline in the expectation of X, denoted

$$\pi = E(X), \tag{1.2}$$

known as the **pure premium**. If it is charged, a company, in the absence of all overhead cost and expenses and all financial income, is in a break-even situation in that it will neither earn nor lose money in the long run; see Appendix ?.

In practice, such pricing is rare, and a company adds a so-called **loading** γ by demanding in the market the premium (or price)

$$\pi^{\mathrm{ma}} = (1+\gamma)\pi, \tag{1.3}$$

so that it actually charges an amount $\gamma\pi$ *above* the break-even situation. We may regard $\gamma\pi$ as the *cost* of risk. In the mathematical notation the distinction between π and π^{ma} will rarely be made visible. Attempts have been made in actuarial literature to determine γ from theoretical considerations; see ?? and ??. This approach is rarely used in practice, and will not be further considered in this book.

A more important question is whether the pure premium actually is known. Where does it come from? In practice mathematical models is used to describe risk. Some simple ones will be employed in Part I and a more systematic study of them will be given in Parts II and III. These arguments always leave unknown parameters, typically determined from experience through statistical estimation or assessed more informally if hard historical data are lacking. Whatever track followed it is an important distinction between the true π characterizing an insurance treaty and the one $\hat{\pi}$ used for analysis and decisions. The issue of **error** in input quantities is a fundamental one; see also Figure 1.1. It applies to all situations and models examined in this book, and we have special notation for it. A ^ over a parameter (or quantity) such as $\hat{\psi}$ means an estimate or assessment of the underlying, unknown ψ. We adhere to this convention throughout. Errors and how we deal with and confront them is taken up in chapter 6.

Portfolios and solvency

The other major issue of insurance risk is **control**. Financial regulators in all countries require insurance companies to set aside enough money to cover their obligations toward the customers. Indeed, this is a major theme in the *legal* definition of insurance. To formulate this mathematically we must introduce the concept of **portfolio**. An insurance company has taken responsibility for many polices. In fact, this was the whole idea in the first place, The company will lose on some (in property insurance those causing damages) and gain on others. With pension schemes the long lives lead to losses (for the company), the short ones to gains. On the portfolio level gains and losses average out. This is the beauty of the idea of a large agent handling many risks simultaneously.

Suppose the portfolio consists of J policies with risk variables $X_1, \ldots, X_J$. Then

$$\mathcal{X} = \sum_{j=1}^{J} X_j \tag{1.4}$$

is the portfolio risk. We shall throughout the book consistently use *caligraphical* letters like $\mathcal{X}$ for quantities applying to portfolios. In (1.4) $\mathcal{X}$ has an expected value that is the sum of all $E(X_j)$. However, with portfolios we are equally often concerned with how far up it fluctuates under unlucky circumstances. In fact, the regulatory offices demand that enough money is reserved to cover $\mathcal{X}$ with very high probability, for example 99% (the percentage vary with the country). In mathematical terms this amounts to solving the equation

$$\Pr(\mathcal{X} > q_\epsilon) = \epsilon \tag{1.5}$$

for the so-called *percentile* q_ϵ. Here ϵ is a small number (for example 1%) and Pr is the probability. The amount q_ϵ set aside to be sufficient to reimburse every claim beyond reasonable doubt is called the **solvency capital** or the **reserve**. A name used for such quanities in finance is Value at Risk (**VaR** for short). As elsewhere there is the problem that it will be a discrepancy (sometimes considerable) between the theoretical q_ϵ we seek and the estimated one $\hat{q}_\epsilon$ we actually use.

1.3 Financial risk: Basic concepts

Introduction

An ordinary bank deposit v_0 grows to $(1+r)v_0$ at the end of one period and to $(1+r)^K v_0$ after K periods. Here r, the **rate of interest**, depends on the length of the time interval. For example, interest compounded over K segments, each of length $1/K$ leads to

$$(1+\frac{r}{K})^K v_0 \rightarrow e^r v_0, \qquad \text{as} \qquad K \rightarrow \infty,$$

after one of the most famous limits in mathematics. It follows that interest *earnings* may be cited as

$$rv_0 \qquad \text{or} \qquad (e^r - 1)v_0$$

depending on how often the "interest on interest" is calculated. The latter leads to nicer formulas with financial derivatives, and will be used there. Of course the two forms are equivalent. We may always raise r slightly to make rv equal to the continously compounded earning.

The purpose of this and the next section is to review extensions (in many directions) of this *risk-free* rate of interest. Gone are the days where actuaries handled liabilities insulated from assets and the companies carried all financial risk themselves. Today there is a growing trend of ceding it back to customers. Indeed, insurance products with financial risk integrated have been sold for decades in countries like Britain and the US under names such as *unit link* or *universal life*. The rationale is that clients receive higher financial income in expectation in exchange for carrying more risk. Pension plans are today increasingly **contributed benfits** (or **CB**) where financial risk rests with the individual members rather than **defined benfits** (or **DB**) where they took none. There is also much interest in using investment strategies taylored to the nature of the liabilities, in particular how they distribute over time. That is known as **asset liability management** (or **ALM** for short), and is discussed in Chapter 13.

What flows from all this is the modern actuary being required to analyse risks of different origin. This is all the more natural to cover in a single book as the basic mathematical, statistical and computational techniques are very much the same so that the user fairly easily can carry them from one area to another. Financial risk is more important in life and pension insurance (which last for decades), but it does enter property insurance too.

Financial returns

Let V_0 be the value of a financial asset at the start of a period and V_1 the value at the end of it. Then

$$R = \frac{V_1 - V_0}{V_0}, \tag{1.6}$$

is called the **return** of the asset. Solving the equation for V_1 yields

$$V_1 = (1 + R)V_0, \tag{1.7}$$

which shows that RV_0 is *financial income* and that R acts like interest. But it is more than that. Interest is a fixed benefit offered by a bank (or a salesman of a very secure bond) in return for making a deposit and is *risk-free.* Shares of company stock, on the other hand, are fraught with risk. They may go up (R positive) or down (R negative). When dealing with such assets, V_1 (and hence R) is determined by the market whereas with ordinary interest r is given and V_1 follows.

The return R is a more general concept than ordinary interest r and is assumed to be random variable following a probability distribution. If random variation is taken away, we are back to a fixed interest r. As r depends on the period of time between V_0 and V_1, so does the distribution of R; how will appear many times in this book. Returns are harder to analyze and work with than interest. The latter follows from the agreement with the bank, whereas the former is unpredicable and can only be described through probability distributions.

Whether interest r really *is* risk-free may not be so obvious as it seems. True, you do get a fixed share of your deposit as reward at the end of each term, but that does not tell its worth in *real* terms. If there is long time horizon, inflation may reduce the value of your contract severely. Indeed, for bonds with long time to maturity the inflationary risk may be considerable. Then there is *opportunity cost* of having entered an agreement at fixed interest when the market spot rate after a while overturns what you get. We discuss and integrate these issues with other sources of risk in Part III of this book.

Log-returns

Economics and finance often construct stochastic models in terms of R directly. An alternative is the **log-return**

$$L = \log(1 + R), \tag{1.8}$$

which by (1.6) can be written as

$$L = \log(V_1) - \log(V_0)$$

as the difference between values on logarithmic scale. The modern theory of financial derivatives, outlined in chapter 13, is based on L. Actually L and R may not be so different since

$$L = R + \frac{R^2}{2} + \frac{R^3}{3} + \ldots,$$

which is the so-called Taylor series of $\log(1+R)$. But R is a rather small quantity and higher powers of R thus become very much smaller than R itself. It follows L deviates little from R over short periods. This will be substantiated in Section 2.3 through a less heuristic argument. When the time horizon is longer, the discrepancy may be larger.

Financial portfolios

In practice investments are often spread on many assets, defining **baskets** or financial **portfolios**. By intuition this must reduce risk; see Chapter 5 where the issue is discussed.. A central quantity is the portfolio **return**, denoted $\mathcal{R}$ (in caligrahical style). Its relationship to the individual returns R_j of the assets is as follows. Let

$$\mathcal{V}_0 = \sum_{j=1}^{J} V_{j0}$$

be the portfolio value at time zero. Here $V_{10}, \ldots, V_{J0}$ are investments in the J assets. At the end of the period the value of the portfolio has grown to

$$\mathcal{V}_1 = \sum_{j=1}^{J} (1+R_j)V_{j0}.$$

Subtract $\mathcal{V}_0$ from $\mathcal{V}_1$ and divide on $\mathcal{V}_0$. This yields *portfolio* return equalling

$$\mathcal{R} = \sum_{j=1}^{J} w_j R_j, \qquad w_j = \frac{V_{0j}}{\mathcal{V}_0}. \tag{1.9}$$

Here w_j is the initial financial **weight** on asset j. Note that

$$w_1 + \ldots + w_J = 1. \tag{1.10}$$

Financial weights define portfolio shares of individual assets and will in this book always satisfy this normalising condition.

The mathematics allow *negative* w_j. Actually this is a practical proposition, since with bank deposits it corresponds to *borrowing*. With shares it is called **short selling** and is a possibility with liquid stocks; i.e. those that are traded regularly in the market (as is the case for most well-known public companies). The loss in a *negative* development is then carried by somebody else. The mechanism is as follows. Our contract with a buyer is to sell shares at the end of the period at an agreed price. At that point we shall have to buy at market price, gaining if the market price is lower than our agreement, losing if not. Short contracts may be an instrument to lower risk; see Chapter 5.

1.4 Risk over time

Introduction

Financial variables are often followed over many periods. When adopting this view, interest rates, returns and values of financial portfolios will be written $\{r_k\}$, $\{R_k\}$ and $\{\mathcal{V}_k\}$, using k

to represent time. Sometimes other variables, such as liabiltities $\{\mathcal{X}_k\}$ will be time-indexed too. The corresponding points in time t_k are always equally spaced; i.e.

$$t_k = kh, \quad k = 0, 1, \ldots, K \tag{1.11}$$

starting at $t_0 = 0$ The terminal point is

$$T = Kh. \tag{1.12}$$

The magnitude of h and T varies enormously with the situation. Life and pension insurance are often concerned with decades. It is then typically sufficient to let h be one year. On the other hand the pricing of financial derivates (Chapter 13) takes h is infitisemally short; i.e. we let $h \to 0$! Continuous time will be used on other occasions too.

Investment strategies
Insurance companies and pension schemes are often concerned with financial risk many periods ahead. If $\mathcal{R}_k$ is portfolio return in period k, the account evolves according to

$$\mathcal{V}_k = (1 + \mathcal{R}_k)\mathcal{V}_{k-1}, \quad k = 1, 2, \ldots, \tag{1.13}$$

where the link of $\mathcal{R}_k$ to the individual assets is through (1.9) as before. In obvious notation we now write

$$\mathcal{R}_k = \sum_{j=1}^{J} w_j R_{kj}. \tag{1.14}$$

An important point is that the weights $w_1, \ldots, w_J$ do not stay the same from one period to another. Individual investments develop differently, changing their relative contribution to the portfolio. When weights above have been written *without* any reference to the time index, it signifies a specific strategy known as **rebalancing**. This means that the portfolio is carefully monitored to keep weights fixed. To do it assets that have fared poorly are bought and the successful ones are sold.

An alternative line of investing is to allow weights to float freely. That strategy is more conveniently expressed mathematically through

$$\mathcal{V}_k = \sum_{j=1}^{J} V_{kj}, \qquad \text{where} \qquad V_{kj} = (1 + R_{kj})V_{k-1,j}, \quad j = 1, \ldots, J.$$

Now individual assets are followed and added to the portfolio value. We would *simulate* the two strategies in slightly different ways, but the stochastic modelling is in terms of returns, as before, see Section 2.4.

Forward interest rates and returns
We often deal with interest rates accumulated over several periods. Consider a bank account v_0 at $t_0 = 0$. At $t_k = kh$ it has grown to

$$(1 + r_k)(1 + r_{k-1}) \cdots (1 + r_1)V_0$$

and the interest rate over the entire period, denoted $r_{0:k}$ is

$$1 + r_{0:k} = (1 + r_1)(1 + r_2)\cdots(1 + r_k). \tag{1.15}$$

If $r_k = r$ is constant, then

$$1 + r_{0:k} = (1 + r)^k.$$

More general notation is sometimes useful, for example $r_{i:k}$ for the **forward** interest rate from t_i to t_k. In this language $r_k = r_{k-1:k}$, reminding us that r_k (and R_k) always apply to the *preceding* period. Similar symbols will be used of returns, for example $R_{0:k}$ for the period from t_0 to t_k

These quantities can't fluctuate freely with respect to each other. That issue will be discussed in Chapter 12. Another point is the following. We do not know what future interest rates are going to be, but our belief on that is crucial for what we are willing to pay for financial instruments like **bonds**; see below. Such assets are traded regularly, and the positions taken reveal a *market view* on interests. In that sense the forward rates $r_{0:k}$ become available for all k. They will be denoted $\hat{r}_{0:k}$ to distinguish them from $r_{0:k}$, the rate that actually appears (which may be very different). The *average* rate $\hat{y}_{0:k}$ per time step is known as the **yield** and is defined by

$$1 + \hat{r}_{0:k} = (1 + \hat{y}_{0:k})^k \qquad \text{or} \qquad \hat{y}_{0:k} = (1 + \hat{r}_{0:k})^{1/k} - 1. \tag{1.16}$$

We are often interested in the entire sequence $\{\hat{r}_{0:k}\}$ as k is varied. This is known as i the **yield curve**[2]. There is a growing trend in contemporary finance for using these implied rates for valuation. That brings us to the next issue.

Present values

What is the value *to-day* of receiving some payment Y_1 at time t_1 if the interest rate is r? Surely it must be $Y_1/(1 + r)$ since that would grow to exactly Y_1 at the end of the period. More generally Y_k at t_k is worth $Y_k/(1 + r)^k$ to-day. This motivates the **present value**

$$\text{PV} = \sum_{k=0}^{K} \frac{Y_k}{(1 + r)^k} \tag{1.17}$$

as a summary of the value of a payment stream $Y_0, \ldots, Y_K$. This criterion is very popular as evaluation criterion in all spheres of economic life. In our applications $Y_0, \ldots, Y_K$ are often described by stochastic models, which means that the present value is stochastic too. Payments could be both positive and negative.

The factor $1+r$ is known as the **discount factor** in that it devaluates or discounts future income to its value if possessed today. With financial derivatives we shall use the continuously compounded version $\exp(r)$ instead. In life insurance r is called the **technical interest rate**, and here we have the weak spot of present values. What technical rate should be

[2] If there are gaps at certain time points due to lack of trading, we might invoke numerical interpolation to fill them in.

used? In insurance it is needed for evaluation decades ahead. Not easy to know interest rates of such time spans! These considerations have motivated the alternative approach of using the market view instead. The present value then becomes

$$\mathrm{PV} = \sum_{k=0}^{K} \frac{Y_k}{1+\hat{r}_{0:k}}, \tag{1.18}$$

where the discount now orginates with the interest rate curve. There is much interest, academic and otherwise, in this approach.

Bonds and yields

One of the most common ways of raising capital for governments and private companies is through **bonds**. In return for money received up-front the issuer makes *fixed* coupon payments at pre-determined time points t_k, $k = 1, 2, \ldots K$, usually with a big amount (known as the **face** of the bond) at expiry at t_K. The coupon payments can be regarded as interest on a loan, but this interpretation is of no significance mathematically. From that point of view bonds are fixed payments at fixed dates. How long they last varies enormously, from a year or less to up to half a century! There is a huge second-hand market for them, and they are traded regularly.

It may seem obvious to value a bond through the present value (1.17), but actually it is the other way a round. The present value is *given* by what the market is willing to pay, and the rate of interest determined by the resulting equation. Thus if $\hat{P}_{0:K}$ is the price traded for the right to the payment stream $\{A_k\}$, its **yield** $\hat{y}$ is the solution of

$$\hat{P}_{0:K} = \sum_{k=0}^{K} \frac{A_k}{(1+\hat{y})^k}. \tag{1.19}$$

With more than one payment a numerical method is needed.

A special case of importance is the **zero-coupon** bond or **T-bond** for which

$$A_0 = \ldots A_{K-1} = 0.$$

The only payment takes place at the maturity t_K of the asset, and in a market operating rationally the zero-coupon bond yield must coincide with yield calculated from the forward rate of interest $\hat{r}_{0:K}$ in (1.16). Bond trading reveals a market the view on future interest (and inflation too). Actually that is how the forward rate is determined but the issue is complicated since there are so many different types. This is dealt with in Part III. To see how the yield curve $\hat{y}_{0:K}$ may look like in practice; consult the model calculations in Figure 6.3 left.

Duration

It is common to measure longevity of bonds and other fixed payment streams through their **duration**. There are in the financial literature several versions . A simple one, based on a fixed technical rate of interest r, is

$$\mathcal{D} = \sum_{k=0}^{K} q_k t_k \qquad \text{where} \qquad q_k = \frac{Y_k(1+r)^{-k}}{\sum_{i=0}^{K} Y_i(1+r)^{-i}}. \tag{1.20}$$

Formally the sequence $\{q_k\}$ is a probability distribution, and duration $\mathcal{D}$ expresses how long the cash flow $\{Y_k\}$ lasts "on average". The interpretaion stems from the "probabiltities" q_k being propertional to the present value of the payment Y_k.

For a zero-coupon bond maturing at $T = t_K = Kh$, we have

$$q_K = 1 \qquad \text{and} \qquad q_k = 0, \ \text{ for } \ k < K.$$

so that $\mathcal{D} = T$, as sensible measure. A bond with fixed coupon payments have duration somewhere between $T/2$ and T. Duration is also a useful concept with liabilities in life insurance; see Chapter 12.

1.5 Examples using Monte Carlo

Introduction

Concepts of risk have been introduced, but no models describing them. This section provides illustrations through very simple means. It is also the intent is indicate how the Monte Carlo method is put to use and its potential power for problem solving, communication and learning. Our tool is the following simple recursion which unifies a number of examples from a computational point of view. For $k = 1, 2, \ldots$ consider

$$Y_k = aY_{k-1} + X_k, \quad k = 1, 2 \ldots, \qquad Y_0 = y_0, \tag{1.21}$$

where a is a parameter. The sequence $\{X_k\}$ are stochastic variables that 'drive' the other sequence $\{Y_k\}$. The only thing we assume about $\{X_k\}$ is *stochastic independence*. This means loosely that random factors and events influencing $\{X_k\}$ in period k bear no relationship to those affecting the sequence in other periods. One possibility for the parameter a is to take $a = 1 + r$, where r is the rate of interest. The recursion then defines the status of an account influenced by random events.

A more advanced specification is

$$a_k = 1 + \mathcal{R}_k, \qquad \text{and} \qquad X_k = -\mathcal{X}_k.$$

Here $a = a_k$ represents financial return, possibly from a large and complicated portfolio whereas X_k are insurance claims that go out of the account. Now $\{Y_k\}$ keeps track on both asset and liability risk. Simulation require one procedure for $\mathcal{R}_k$ and one for $\mathcal{X}_k$, integrated through Algorithm 1.1 below. Examples of this nature will be discussed in Chapter 13.

A skeleton algorithm

The simulation of $\{Y_k\}$ is carried out by the following skeleton scheme, which introduces the first algorithm of the book:

Algorithm 1.1 Basic recursion

0 Input: y_0, a.
1 $Y_0^* \leftarrow y_0$ *%Initialisation*
%Draw K here if random

2 For $k = 1, \dots, K$ do
3 Sample X_k^* %*Many possibilities*
4 $Y_k^* \leftarrow aY_{k-1} + X_k^*$ %*New value*

5. Return $Y_0^*, \dots Y_K^*$ (or just Y_K^*)

We start by initializing (step 1), and then successively draws the random terms X_k^* (step 3) that revise the previous values Y_{k-1}^*. Note that all sampled variables are $*$-marked. The backward arrow $\leftarrow$ signifies that the variable on the left is assigned the value on the right. It is a more convenient notation than an ordinary equality sign, as will emerge later[3]. The % symbol will be used throughout the book to insert comments. Actually most simulation experiments in insurance and finance fit this scheme or some simple variation of it.

The recursion (1.21), implemented through Algorithm 1.1, unites many of the basic models in insurance and finance. Here are four examples.

Insurance portfolios
Consider a portfolio of J insurance policies and suppose the sum of all compensations to customers is to be evaluated for the following year, uncertainty included. One approach is to draw all X_j randomly and add them together. That is exactly what is achieved in Algorithm 1.1 if $a = 1$ and $y_0 = 0$, provided, of course, that liabilities are generated appropriately.

The example shown in Figure 1.2 is for so-called **term** insurance, where there is a one-time payment to a beneficiary upon the death of the policy holder. Imagine that the insured sums ζ_j are stored on file. We would also have access to age and sex of the policy holders, from which their probability ${}_1p_j$ of surviving the coming year can be inferred; see Section 3.4 for details. Note the notation ${}_1p_j$ which is indigeneously actuarial; we'll meet more of that in Chapter 10. It follows that the liability model is

$$\Pr(X_j = 0) = {}_1p_j, \qquad \Pr(X_j = \zeta_j) = 1 - {}_1p_j,$$

and it is a simple matter to go through the portfolio, read policy information from file, draw *randomly* those who die (whom we have to pay for) and add all payments together.

The example in Figure 1.2 were run with 10000 policies for which the sum insured was one million US$ for each. Survival probabilities and the age distribution of the policy holdes were as specified in Section 3.4. The insured were between 30 and 60 years. One hundred parallel runs through the portfolio are plotted jointly on the left showing how the simulations develop. The curved shape has no significance. It is due to the age of the policy holders having been ordered on the file so that the young ones with low death rates are tested first.

[3]For example, if we only want the *last* value Y_K^*, as is frequent, we may write statements like $Y^* \leftarrow aY^* + X^*$ overwriting Y^*; past values *not* being stored in the computer. Many of the algorithms in the book will be presented in this way.

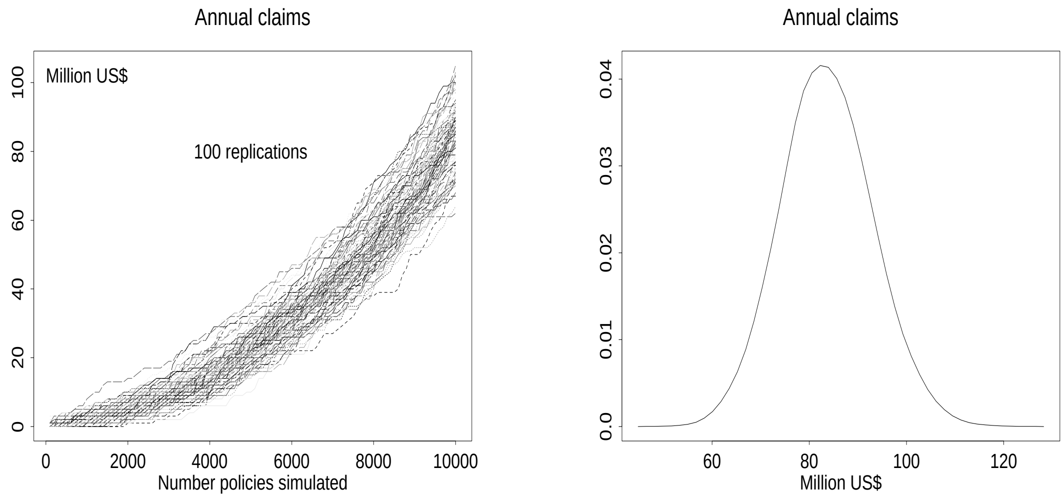

Figure 1.2 Simulations of term insuarnce. Left: 100 *parallel runs through insurance portfolio. Right: Annual density function.*

What counts is the variation between (say) 70 and 100 million US \$ after having gone through the entire portfolio. On the right of Figure 1.2 this variation is converted into an estimated probability density function. For that 10000 simulations were used (and the so-called kernel density estimator explained in Section 2.2). The Gaussian looking shape follows from the central limit theorem[4]. In life insurance this kind of risk is often ignored. You will see why in Section 3.4.

Insurance portfolios of identical risks
A common model in property insurance (not the least in textbooks) is that of *identical* risks. Claims then appear on average equally often for all policies, and there is no *systematic* variation in their cost. The portfolio then has to cover

$$\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$$

where $Z_1, Z_2, \ldots$ are the payments and $\mathcal{N}$ their number. We no longer have to keep track on which policy a claim come from since the probability distribution is the same anyway. There will be more on this representation in Section 3.2.

Simulation algorithms have much in common with the preceding example. The principal difference is that claim frequency $\mathcal{N}^*$ must generated *before* entering the for-loop in Algorithm 1.1 (as indicated there). Subsequently $\mathcal{N}^*$ damages are drawn and added. Details are recorded as Algorithm 3.1 in Chapter 3.

The example in Figure 1.3 was run with annual claim frequency 1% per policy, assuming the standard Poisson model (Section 2.6 and Chapter 7). Claim size was taken from the so-called *Danish fire data* (Section 8.2). This is an historical material comprising over 2000

[4]You need the Lindeberg extension of this celebrated result; see ??.

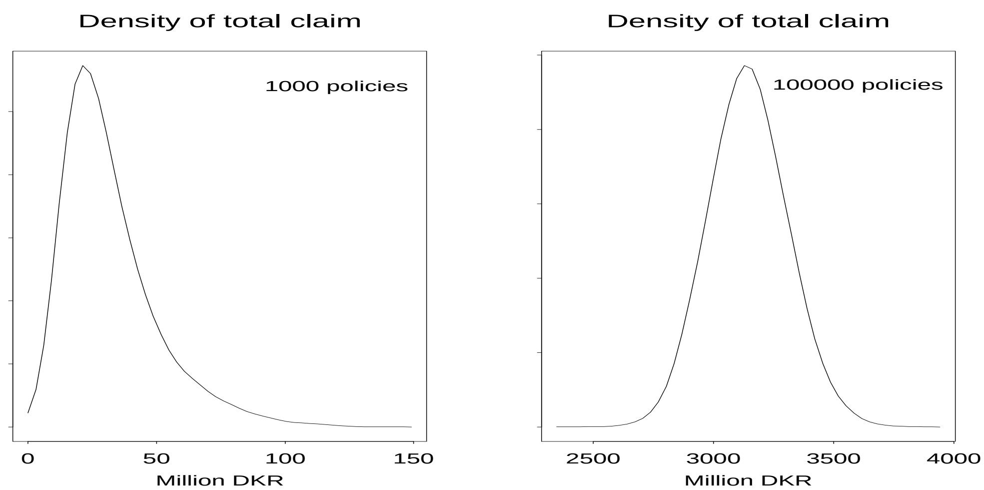

Figure 1.3 Density functions of the total claim against portfolio of fire risks (eight Danish kroner (DKR) is about one euro).

fires, the largest one going up to several hundred million Danish kroner (DKR)[5]. All those on record was considered equally likely to re-appear in the future, an approach discussed in Section 8.2. Evaluations of the probability distribution of portfolio payments are shown in in Figure 1.3 for a "small" portfolio ($J = 1000$) on the left and a "large" one ($J = 100000$) on the right. The density estimates were in both cases calculated from 10000 simulations. Random variation is more important than in the preceding example. For the small portfolio the density is skewed to the right, which implies a substantial likelihood of very much larger losses. If you consult Figure 8.1, you will discover the same shape in the Danish fire data. As portfolio size grows, the asymmetry is straightened out and the distribution becomes Gaussian, as predicted by the central limit theorem

Reversion to mean

Simulation is also useful to understand (with minimal mathematics) the behaviour of stochastic models for financial variables. Consider first quantities like interest rate, volatility, rate of inflation and exchange rates. All those tend to fluctuate between certain, not clearly defined limits. If they swing to far out on either side, there are forces in the economy that tend to drag them back again. This is known as *reversion to mean* and plays an important role in the mathematical description of all the quantities mentioned.

Consider interest rates. The most widely used model in finance is due to **Vasiček** (1977) and bears his name. Elsewhere it is usually called **first order autoregressive** and is then written

$$\begin{aligned} r_k &= Y_k + \xi \\ Y_k &= aY_{k-1} + \sigma\varepsilon_k, \qquad Y_0 = r_0 - \xi. \end{aligned} \tag{1.22}$$

[5]There is about eight DKR in one euro.

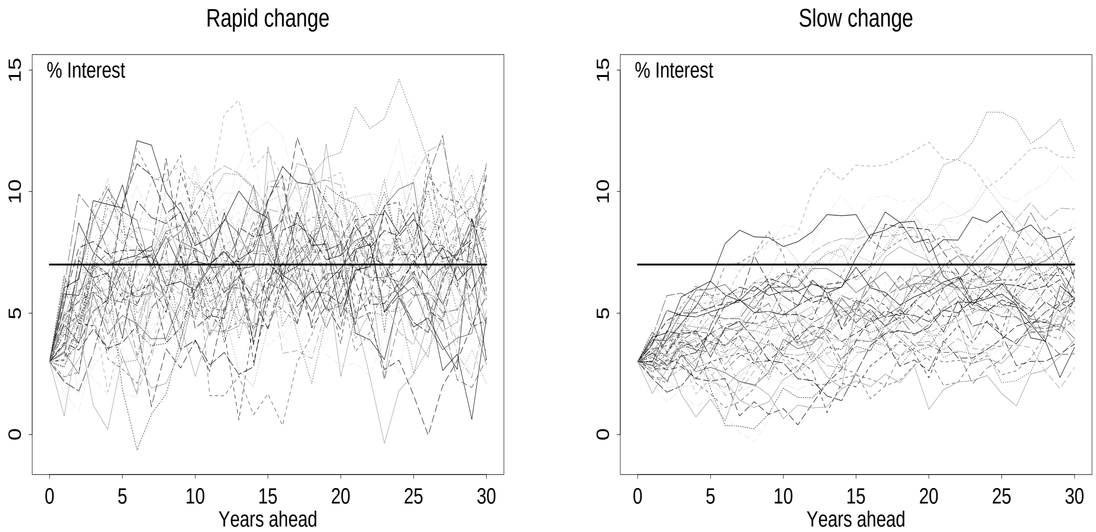

Figure 1.4 Simulations of the annual rate of interest from the Vasiček model.

Here ξ, a and σ are fixed parameters, and $\{\varepsilon_k\}$ consist of independent terms, all having the same probability distribution with mean 0 and standard deviation 1[6]. The series $\{Y_k\}$ are simulated by Algorithm 1.1 and ξ is added to produce $\{r_k\}$. In Figure 1.4 all random terms are Gaussian.

Simulation experiments were run under two different model scenarios of annual parameters:

Rapid change:	$r_0 = 3\%$	$\xi = 7\%$	$a = 0.70$	$\sigma = 0.016$
Slow change:	$r_0 = 3\%$	$\xi = 7\%$	$a = 0.95$	$\sigma = 0.007$
	Annual parameters			

In either case is the value in the beginning (at 3%) much lower than the long-term average (7%) around which the interest rate eventually fluctuates. That level is quickly reached in the first model scenario (Figure 1.4 left), and there is from then on no *systematic* change in the oscillations. Such phenomena are called **stationary**, and they are discussed in Chapter 12. Even the second model scenario would eventually produce such behaviour, but it takes much longer time (because a is close to one). After 30 years the movements is still slightly on the rise on average. Values of a approaching one from below produce smoother, less volatile developments. When $a = 1$ the character of the model changes completely. That is dealt with next.

Perfect markets: Equity

Stock prices $\{S_k\}$ are *not* mean reverting. Why not? Because unlike interest and inflation, they are traded commodities. If we knew of a *systematic* factor tending to drive them up or down, we would be able to act upon it and earn money. But that opportunity would be

[6] That convention is followed throughout. Series denoted $\{\varepsilon_k\}$ always satisfy these conditions.

available to everybody, making the idea useless. Another way to put it is that the market would allow *arbitrage*, i.e. risk-less financial income. This notion, underlying the pricing of modern financial derivatives, will be introduced in chapter 13.

The standard model for equity returns is

$$R_k = \exp(\xi + \sigma\varepsilon_k) - 1, \qquad k = 1, 2 \ldots. \tag{1.23}$$

Here ε_k (as above) is a sequence of independent random variables with mean zero and standard deviation one. By definition the value of the stock then evolve according to

$$S_k = (1 + R_k)S_{k-1},$$

which after inserting for R_k becomes

$$S_k = \exp(\xi + \sigma\varepsilon_k)S_{k-1} \quad k = 1, 2 \ldots, \qquad S_0 = s_0, \tag{1.24}$$

which might be called a **geometric random walk**. On logarithmic scale $Y_k = \log(S_k)$ we have an ordinary random walk; see (1.25) below. In continuous time as $h \to 0$ the model is in mathematical finance called **geometric Brownian motion**.

The model is simulated in Figure 1.5, but in the form of the financial returns $R_{0:k}$ rather than the share price directly. The intial value s_0 then drops out, and we are lead to the scheme

$$\begin{aligned} R_{0:k} &= \exp(Y_k) - 1 \\ Y_k &= Y_{k-1} + \xi + \sigma\varepsilon_k, \quad k = 1, 2, \ldots \qquad Y_0 = 0 \end{aligned} \tag{1.25}$$

Simulation is again a tiny variation of Algorithm 1.1.

The parameters chosen were on monthly time scale:

Low yield and risk:	$\xi = 0.4\%$	$\sigma = 4\%$
High yield and risk:	$\xi = 0.8\%$	$\sigma = 8\%$
	Monthly parameters	

These specifications could be representative for single equities. What emerges in Figure 1.5 is a behaviour entirely different from that of interest rates. Mathematically that is due to the the coefficient a which was 0.7 and 0.95 in Figure 1.4 and 1 now. The crucial detail is whether a is *less* then one or *equal* to one (more in Chapter 12). There is in Figure 1.5 strong potential for *huge* gain and also for *huge* loss. Up to 50% of the orginal capital is lost in a few of the simulations on the right! Be aware that scales on the vertical axes are not the same. In reality the first model scenario returns a spread than is (perhaps) one third of the other. Risk *reduction* seems called for. One possibility is portfolios of *different stock*. The effect of that will be examined in Chapter 3.

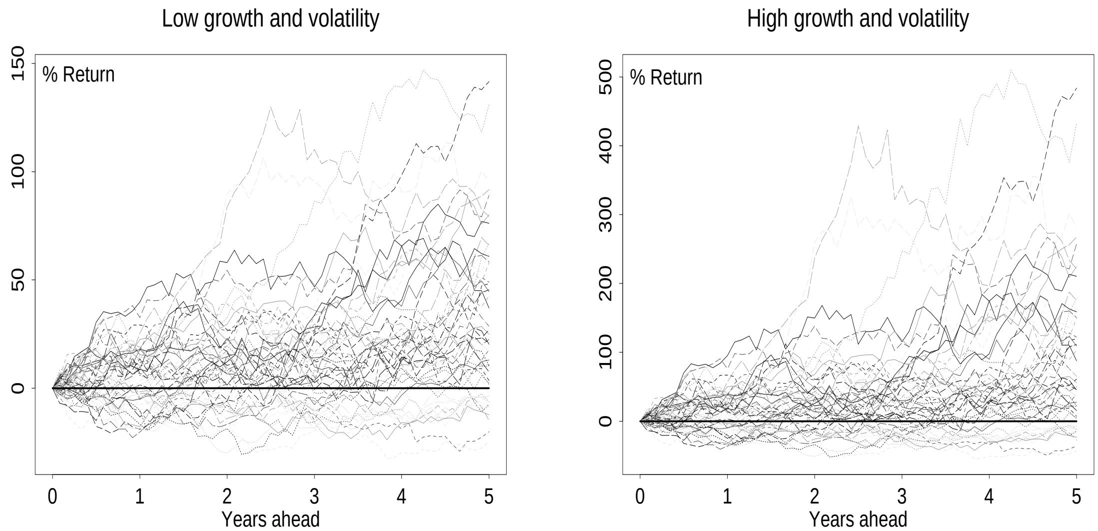

Figure 1.5 Simulations of accumulated equity return from geometric random walk (monthly scale).

1.6 Outline of the book

The first part of the book is an introduction to the basic tools of quantitative risk analysis. An elementary course in probability and statistics is taken for granted (although a brief appendix is included), but such courses rarely move very far into the notion of *dependence* between stochastic variables. That issue is the key to most interesting risk modelling. An introduction is given in Chapter 5. Another issue which should be part of our basic thinking, is *error*. Models underlying statements of risk are often grossly inaccurate. That we must live with, but we should at least understand what it implies. One consequence is that simulation uncertainty often is minor and completely over-shaddowed by contributions from other sources. Chapter 6 is an introduction to error in risk analysis.

But Part I of this book is above all on the Monte Carlo method and how its potential is exploited. That is a question of *techniques* (Chapters 2 and 4), but also of a line of *thinking*. For much problem solving we do not need complicated probabilistic descriptions of the stochastic models involved. All that is necessary is how they are simulated in the computer! Such an approach requires much less of mathematical prerequisites, and we shall be able to get quicker to more "advanced" material. Chapter 3 will demonstrate how a wide range of problems, coming from all areas of actuarial and financial risk, is solvable by writing small (and simple) simulation procedures. Part of the skill (and a rewarding way to work) is efficient use of the computer, notably the merging of computer programs with others to handle increasingly more complex situations and reuse with adjustments to deal with related, but different problems.

The two other parts of the book are more systematics treatments of risk topics, using the machinery from Part I as a tool and with more on the probabilistic side of modelling.

General insurance is considered in Part II, but without investment which have a more natural home with all the other methods that draw on stochastic processes. That is the subject of Part III which deals with life insurance and financial risk. There is a Chapter on financial derivatives and arbitrage pricing which do *not* rely on stochastic calculus.

1.7 Further reading