Exercises

Section 10.2

Exercise 10.2.1 Consider a portfolio of identical risks where the standard deviation of the claim size model is $\sigma_z = \theta\xi_z$ where θ is a parameter. **a)** Show that the normal approximation for the reserve can be written

$$q_\epsilon^{\mathrm{No}} = E(\mathcal{X})(1+\gamma) \qquad \text{where} \qquad \gamma = \sqrt{\frac{1+\theta^2}{\mu TJ}}\,\phi_\epsilon.$$

Let Z be Gamma-distributed so that $\theta = 1/\sqrt{\alpha}$ where α is the shape parameter. **b)** Compute γ when $\mu = 5\%$, $T = 1$, $\alpha = 1$, $\epsilon = 1\%$ (so that $\phi_\epsilon = 2.33$) and $J = 100$ and $J = 10000$. For which of the two values of J is the approximation most reliable?

Exercise 10.2.2 This is an extension of the preceding exercise. **a)** Show that the normal power approximation can be expressed as

$$q_\epsilon^{\mathrm{NP}} = E(\mathcal{X})(1+\gamma) \qquad \text{where} \qquad \gamma = \sqrt{\frac{1+\theta^2}{\mu TJ}}\,\phi_\epsilon + \frac{(\zeta_z\theta^3 + 3\theta^2 + 1)(\phi_\epsilon^2 - 1)}{6(1+\theta^2)\mu TJ}$$

For a Gamma distribution $\theta = 1/\sqrt{\alpha}$ and $\zeta_z = 2/\sqrt{\alpha}$. **b)** Insert those into the preceding expression for γ. **c)** Investigate the impact of the NP-term on γ numerically under the same conditions as in Exercise 10.2.1b).

Section 10.3

Exercise 10.3.1 For a portfolio of identical risks suppose $Z = \xi_z G$ where $G \sim \mathrm{Gamma}(\alpha)$. **a)**. Compute the normal and normal power approximation to the reserve at level $\epsilon = 1\%$ when $\alpha = 1$ and $J\mu = 10$, 100 and 1000. **b)** Repeat the computations in a) by means of simulations using $m = 10000$. **c)** Compare the results in a) and b) and comment on how the discrepancies depend on $J\mu$.

Exercise 10.3.2 Suppose you want to plan a simulation experiment for the reserve so that Monte Carlo error is less than a certain fraction γ of the final result. One strategy is to run B batches of m_1 simulations. For each batch b sort the simulations in descending order and compute $q_{\epsilon b}^* = \mathcal{X}_{(m_1\epsilon)}^*$ as the reserve. That gives you B assessments $q_{\epsilon 1}^*, \ldots, q_{\epsilon B}^*$ from which you may compute their mean $\bar{q}_\epsilon^*$ and standard deviation s_ϵ^*. **a)** Carry out $B = 10$ rounds of such experiments when $J\mu = 10$ and Z is exponentially distributed with mean one. Use $\epsilon = 1\%$ and $m_1 = 10000$. **b)** Compute mean and standard deviation $\bar{q}_\epsilon^*$ and s_ϵ^* of $q_{\epsilon 1}^*, \ldots, q_{\epsilon B}^*$. For large m there is the approximation $\mathrm{sd}(q_\epsilon^*) \doteq a_\epsilon/\sqrt{m}$ where a_ϵ doesn't depend on m; see Section 2.2. **c)** Estimate a_ϵ as the value $a_\epsilon^* = s_\epsilon^*\sqrt{m_1}$ and argue that

$$m = m_1\left(\frac{s_\epsilon^*}{\gamma\bar{q}_\epsilon^*}\right)^2$$

is approximately the number of simulations you need. **d)** Compute it for the values you found in b) when $\gamma = 1\%$.

Section 10.4

Exercise 10.4.1 A quick way to explore statistical significance is the **Wald** test. Let $\hat{\theta}$ be an estimate of a parameter θ and $\hat{\sigma}_\theta$ its estimated standard deviation. Then pronounce the underlying θ different from zero if $|\hat{\theta}/\hat{\sigma}_\theta| > 2$. The significance level is close to 5% under the normal approximation which is a fair assumption in many applications of regression methodology. **a)** Apply this test to the second age category in Table 10.4. Can we from this information be sure that age has real impact on claim frequency and size? **b)** Examine the two other explanatory variables in Table 10.4 (distance limit and geographical region) in the same way. Which of the categories deviate significantly from the first one?

Exercise 10.4.2 The estimate of the pure premium for a customer with a given set of explanatory variables is $\hat{\pi} = Te^{\hat{\eta}}$ where mathematical expressions for $\hat{\eta}$ and its estimated standard deviation $\hat{\tau}$ were given in Section 10.4. Let $\underline{\pi} = \hat{\pi}e^{-2\hat{\tau}\phi_\epsilon}$ and $\bar{\pi} = \hat{\pi}e^{2\hat{\tau}\phi_\epsilon}$ where ϕ_ϵ is the $1-\epsilon$ percentile of the standard normal distribution. **a)** Argue using the normal approximation that $\underline{\pi} < \pi < \bar{\pi}$ is $1-2\epsilon$ confidence interval for π. **b)** Compute 95% condifence intervals for the pure premia in Table 10.5 utilizing that $\phi_\epsilon \doteq 2$ when $\epsilon = 2.5\%$.

Exercise 10.4.3 Consider a portfolio where regression models for claim intensity and claim size have been fitted. Then $\log(\mu_j) = b_{\mu 1}x_{j0} + \ldots + b_{\mu v}x_{jv}$ and $\log(\xi_{zj}) = b_{\xi 1}x_{j0} + \ldots + b_{\xi v}x_{jv}$ are known relationships for policy holder j. If $\sigma_{zj} = \sigma_z$ is the common standard deviation for all j, use the central limit theorem to compute the approximate reserve for the portfolio at level ϵ.

Section 10.5

Exercise 10.5.1 Consider the linear credibility estimate $\hat{\pi}_K = (1-w)\zeta + w\bar{X}_K$ where $w = \upsilon^2/(\upsilon^2 + \tau^2/K)$ and $\zeta = E\{\pi(\omega)\}$, $\upsilon^2 = \text{var}\{\pi(\omega)\}$ and $\tau^2 = E\{\sigma^2(\omega)\}$ are the three structural parameters. Explain and interprete why the credibility set-up yields a weight w which is increasing in K and υ and decreasing in τ.

Exercise 10.5.2 The accuracy of the group credibility estimate $\hat{\Pi}_K$ may be examined by calculating relative error. **a)** Use the standard deviation formula (**??**) to verify that

$$\frac{\text{sd}(\hat{\Pi}_K - \Pi)}{E(\Pi)} = \frac{\upsilon/\zeta}{\sqrt{(1 + KJ\upsilon^2/\tau^2}},$$

b) What happens to the ratio as $J \to \infty$? Explain what this tells us about the accuracy of credibility estimation on group level.

Exercise 10.5.3 Exercise 10.5.2 enables us to re-analyse the accuracy of credibility estimation. The following conditions are those of Table 10.7 with μ being a common, random factor influencing the entire portfolio. Then, $\zeta = \xi_\mu \xi_z$, $\upsilon^2 = \sigma_\mu^2 \xi_z^2$ and $\tau^2 = \xi_\mu(\xi_z^2 + \sigma_z^2)$ if $T = 1$; see Section 10.5. Suppose $\mu = 5.6\%$, $\sigma_\mu = 2\%$, and $\sigma_z = 0.1\xi_z$ as in Table 10.7. **a)** Show that the standard deviation/mean ratio in Exrecise 10.5.2 doesn't depend on ξ_z. **b)** Compute it for $K = 0$, 10 and 20 both when $J = 1$ (single polices) and when $J = 10000$. **c)** What conclusions do you draw from these computations?

Exercise 10.5.4 The optimal credibility estimate $\hat{\pi}_K = \zeta(\bar{n} + \alpha/K)/(\xi_\mu T + \alpha/K)$ is an adjustment of the average, pure premium ζ of the population. Clearly $\hat{\pi}_K > \zeta$ if $\bar{n} > \xi_\mu T$ and $\hat{\pi}_K \leq \zeta$ in the opposite case. **a)** What is the intuition behind this? **b)** Show that the adjustment increases with K and decreases with α. Why must the credibility set-up lead to these results?

Exercise 10.5.5 Let $\Pi = E(\mathcal{X}|\mu)$ be the average claim against a portfolio when claim intensity μ is a common random factor with prior distribution $\mu = \xi G$ with $G \sim \text{Gamma}(\alpha)$. Suppose $\bar{n}$ is the average number of claims against the portfolio over K years. **a)** Explain why $\hat{\Pi}_K = J\zeta(\bar{n}+\alpha/K)/(J\xi_\mu T+\alpha/K)$ is the optimal credibility estimate of Π. **b)** Use the standard deviation formula (**??**) right to deduce that

$$\frac{\text{sd}(\hat{\Pi}_K - \Pi)}{E(\Pi)} = \frac{1}{\sqrt{\alpha + J\xi_\mu KT}}.$$

c) What is the limit as $J \to \infty$? Comment on the potential of optimal credibility estimation on group level.

Section 10.6