STK 4020: Bayesian Statistics
Autumn 2010

Course Notes and Exercises
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1. Prior to posterior updating with Poisson data

This exercise illustrates the basic prior to posterior updating mechanism for Poisson data.

(a) First make sure that you are reasonably acquainted with the Gamma distribution. We say that $Z \sim \text{Gamma}(a, b)$ if its density is

$$g(z) = \frac{b^a}{\Gamma(a)} z^{a-1} \exp(-bz) \quad \text{on } (0, \infty).$$

Here a and b are positive parameters. Show that

$$\text{E}\, Z = \frac{a}{b} \quad \text{and} \quad \text{Var}\, Z = \frac{a}{b^2} = \frac{\text{E}Z}{b}.$$

In particular, low and high values of b signify high and low variability, respectively.

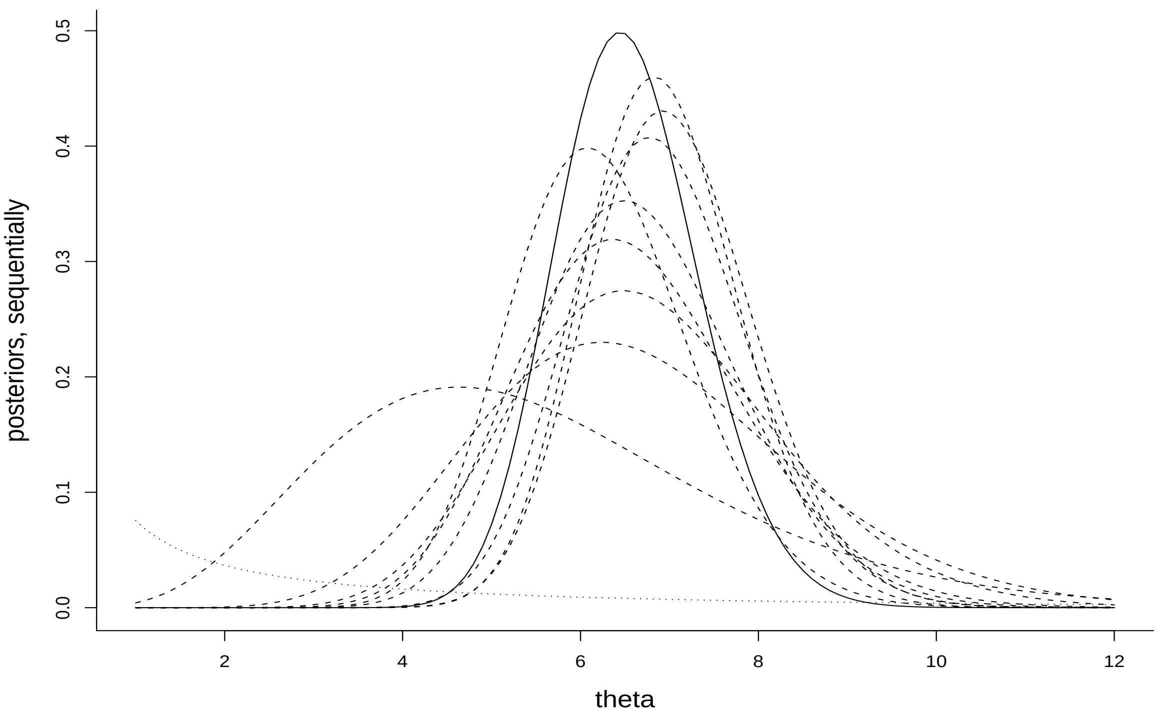

Figure 1: Eleven curves are displayed, corresponding to the $\text{Gamma}(0.1, 0.1)$ *intial prior density for the Poisson parameter* θ *along with the ten updates following each of the observations 6, 8, 7, 6, 7, 4, 11, 8, 6, 3.*

(b) Now suppose $y \mid \theta$ is a Poisson with parameter θ, and that θ has the prior distribution $\text{Gamma}(a, b)$. Show that $\theta \mid y \sim \text{Gamma}(a + y, b + 1)$.

(c) Then suppose there are repeated Poisson observations $y_1, \ldots, y_n$, being i.i.d. $\sim \text{Pois}(\theta)$ for given θ. Use the above result repeatedly, e.g. interpreting $p(\theta \mid y_1)$ as the new prior before observing y_2, etc., to show that

$$\theta \mid y_1, \ldots, y_n \sim \text{Gamma}(a + y_1 + \cdots + y_n, b + n).$$

Also derive this result directly, i.e. without necessarily thinking about the data having emerged sequentially.

(d) Suppose the prior used is a rather flat $\text{Gamma}(0.1, 0.1)$ and that the Poisson data are 6, 8, 7, 6, 7, 4, 11, 8, 6, 3. Reconstruct a version of Figure 1 in your computer, plotting the ten curves $p(\theta \mid \text{data}_j)$, where data_j is $y_1, \ldots, y_j$, along with the prior density. Also compute the ten Bayes estimates $\widehat{\theta}_j = \text{E}(\theta \mid \text{data}_j)$ and the posterior standard deviations, for $j = 0, \ldots, 10$.

(e) The mathematics turned out to be rather uncomplicated in this situation, since the Gamma continuous density matches the Poisson discrete density so nicely. Suppose instead that the initial prior for θ is a uniform over $[0.5, 50]$. Try to compute posterior distributions, Bayes estimates and posterior standard deviations also in this case, and compare with you found above.