UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4050/9050 — Statistical simulations and computation.

Day of examination: Thursday December 8th 2011.

Examination hours: 09.00 – 13.00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

(a) For a Weibull distributed random variable $T > 0$ we may write the survival probability

$$\mathrm{P}(T > t) = \exp(-\theta t^{\alpha})$$

where $\theta > 0$ and $\alpha > 0$ are parameters. Derive the inverse transform method for simulating Weibull distributed random variables.

(b) Assume that Y is a gamma-distributed variable with shape parameter equal to 2 and a rate parameter λ, so that Y has density $g(y) = \lambda^2 y \exp(-\lambda y)$.

Explain why the inverse transform is more difficult in this case.

Present two approaches to simulating random variables from this distribution.

(c) Assume that (T, θ) has simultaneous density

$$g(t, \theta) = \theta \alpha t^{\alpha - 1} \exp(-\theta t^{\alpha}) \lambda \exp(-\lambda \theta)$$

Explain why we then have that T conditional on θ is Weibull with parameters α and θ as parameterized in question (a).

What is the marginal (unconditional) distribution of θ?

(d) Derive the conditional distribution of θ in question (c) given that the random variable $T = t$.

Present the Gibbs sampling algorithm for sampling from the bivariat distribution $g(t, \theta)$ in question (c).

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Problem 2

Assume that we want to evaluate an integral (expectation) $l = \int H(x)f(x)dx = \mathrm{E}_f[H(X)]$ where $H(x)$ is some function, and X a random variable with density $f(x)$.

(a) Argue that if $g(x)$ is a density such that $g(x) > 0$ if $H(x)f(x) > 0$ then we may estimate l by sampling $Y_i, i = 1, \ldots, n$ independently from $g(x)$ and calculating the average $\hat{l} = \frac{1}{n}\sum_{i=1}^{n} H(Y_i)f(Y_i)/g(Y_i)$.

State an expression for the variance of $\hat{l}$.

(b) Assume that $H(x) \geq 0$. Argue that the optimal density $g(x)$ to sample from satisfies $g(x) \propto H(x)f(x)$.

Discuss also why this result is not directly useful in practice, but may still give suggestions for sampling distributions $g(x)$.

Furthermore, extend the optimality result to functions $H(x)$ not necessarily non-negative.

(c) Assume that X has density $f(x) = C\exp(-x^{1.5})$ for $x > 0$ and $f(x) = 0$ for $x < 0$ (where approximately $C = 1.1077325$). We want to estimate the probability $X > 5$. Why is it not a good idea to simply sample n random variables X_i from $f(x)$ and calculating the proportion > 5?

Importance sampling like in question (a) is a better idea. Suggest a sampling density $g(x)$ that may give a quite precise estimate.

(d) We may instead be interested in calculating $E_f[X^2]$. Suggest a sampling density $g(x)$ suitable for this problem.

Why would a good solution to question (c) be a poor choice in this case?

Problem 3

Assume that $X_0, X_1, X_2, \ldots$ is a time-homogeneous Markov chain on a finite state space $\{1, 2, \ldots, m\}$ with transition probabilities $p_{ij} = P(X_n = j | X_{n-1} = i)$. Assume also that the Markov chain is aperiodic and irreducible.

(a) Assume that for some values $\pi_1, \pi_2, \ldots, \pi_m$ the Markov chain satisfies the detailed balance equations

$$\pi_i p_{ij} = \pi_j p_{ji}$$

Explain why the π_i defines the stationary distribution for X_n and why it will also be the limit distribution for X_n when $n \to \infty$.

The equation $\pi_i p_{ij} = \pi_j p_{ji}$ is also referred to as reversibility. Give an explanation for this term.

(b) The Metropolis-Hastings algorithm generates Markov chains given as follows: From current state of the chain $X_n = i$

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(i) Sample a proposal value $Y = j$ from a conditional distribution $q(y|X_n = i)$

(ii) Accept Y, i.e. let $X_{n+1} = j$ with probability

$$\alpha_{ij} = \min(1, \frac{\pi_j q(i|j)}{\pi_i q(j|i)})$$

(iii) Reject Y, i.e. let $X_{n+1} = i$ with probability $1 - \alpha_{ij}$.

Show that the Metropolis-Hastings algorithm satisfies the detailed balance equation and conclude about how one may sample from the distribution given by the π_i from the algorithm.

Discuss properties that the proposal distribution should have for the sampling from the stationary distribution to be efficient.

END