UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	STK4050 — Statistical simulations and computation
Day of examination:	FASIT
Examination hours:	14.30 – 17.30.

This examination set consists of 3 pages.

Appendices:	None.
Permitted aids:	Accepted calculator.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

(a) Generate $X = G^{-1}(U)$ where $U \sim \text{uniform}[0,1]$. Then

$$\Pr(X \leq x) = \Pr(G^{-1}(U) \leq x) = \Pr(U \leq G(x)) = G(x)$$

showing that X has the right cummulative distribution function. Solving $G(x) = u$ gives $G^{-1}(u) = -\log(u^{-1} - 1)$.

(b)

$$h(x) = -0.5\log(2\pi) - 0.5x^2 + x + 2\log(1 + e^{-1})$$

giving

$$h'(x) = -x + 1 - 2\frac{e^{-x}}{1 + e^{-x}} = -x + 1 - 2\frac{1}{1 + e^{x}}$$

which is equal to 0 for x=0. Further,

$$h''(x) = -1 + 2\frac{e^{x}}{1 + e^{x}}$$

which is always negative.

(c) Define $M = \max_x f(x)/g(x) = f(0)/g(0) = 4/\sqrt{2\pi}$. Then generate $x \sim g(\cdot)$ and accept with probability $f(x)/[Mg(x)]$.

The acceptance rate is $M^{-1} = 0.6267$.

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Problem 2.

(a) Main idea: Generate sequence $\boldsymbol{x}^1, \boldsymbol{x}^2, ...$ such that $\boldsymbol{x}^s$ converges (in distribution) to $f(\cdot)$.

Requirement: f is the invariant distribution, irreducible and aperiodic chain.

Strengths: General algorithm, can be used on very complex problems.

Weaknesses: Approximate method, only right distribution in the limit. Need to find a burnin such that we get approximately right distribution. We also get dependent variables which increase the variance of our Monte Carlo estimates.

(b) Systematic scan:

- Generate $y_1^s \sim N(x_1^{s-1}, d_1^2)$
- Put $x_1^s = y_1^s$ with probability $\min\{1, f(y_1^s, x_2^{s-1})/f(x_1^{s-1}, x_2^{s-1})\}$, otherwise $x_1^s = x_1^{s-1}$. (Note that no acceptance if $y_1^s < 0$. One can use alternative proposals taking the constraint on x_1 into account.)
- Generate $y_2^s \sim N(x_2^{s-1}, d_2^2)$
- Put $x_2^s = y_2^s$ with probability $\min\{1, f(x_1^s, y_2^s)/f(x_1^s, x_2^{s-1})\}$, otherwise $x_2^s = x_2^{s-1}$.

Random scan: Select which of x_1 and x_2 to change.

Simultaneous updating:

- Generate $y_j^s \sim N(x_j^{s-1}, d_j^2), j = 1, 2$
- Put $(x_1^s, x_2^s) = (y_1^s, y_2^s)$ with probability $\min\{1, f(y_1^s, y_2^s)/f(x_1^{s-1}, x_2^{s-1})\}$. (Note again that no acceptance if $y_1^s < 0$. Can use alternative proposals taking the constraint on x_1 into account.)

(c) We have

$$\begin{aligned}
f(x_1|x_2) \propto & f(x_1, x_2)\\
& \propto \exp\left[-x_1 0.5(1 + x_2^2) +\right]\\
& \propto 0.5(1 + x_2^2) \exp\left[-x_1 0.5(1 + x_2^2)\right]
\end{aligned}$$

showing that $x_1|x_2$ is exponential distributed with parameter $0.5(1 + x_2^2)$. Further

$$\begin{aligned}
f(x_2|x_1) \propto & f(x_1, x_2)\\
& \propto \exp\left[-0.5 x_1(1 + x_2^2)\right]\\
& \propto \frac{\sqrt{x_1}}{\sqrt{2\pi}} \exp\left[-0.5 x_1 x_2^2\right]
\end{aligned}$$

showing that $x_2|x_1$ is Gaussian with expectation zero and variance $1/x_1$. The Gibbs sampler then goes as follows:

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- Generate $x_1^s \sim \exp(0.5(1 + (x_2^s)^2))$.
- Generate $x_2^s \sim N(0, 1/x_1)$.

(d) Gibbs sampler requires more work in the sense that the conditional distributions need to be worked out. On the order hand, there is no need to choose proposal distributions or tuning parameters.

Problem 3.

(a) The density $p(x_t|\boldsymbol{y}_{1:t})$ is difficult to evaluate, while

$$p(\boldsymbol{x}_{1:t}|\boldsymbol{y}_{1:t}) \propto p(\boldsymbol{x}_{1:t})p(\boldsymbol{y}_{1:t}|\boldsymbol{x}_{1:t})$$

which is easy to evaluate.

(b) The weights being equal to one follows directly from the definition of

$$\begin{aligned} w_1(x_1) =& \frac{p(x_1|\boldsymbol{y}_1)}{q_1(\boldsymbol{x}_1|y_1)} \\ =& \frac{p(x_1|\boldsymbol{y}_1)}{p(\boldsymbol{x}_1|y_1)} = & 1 \end{aligned}$$

Then all samples have equal weight and we get a high effective sample size.

(c) A weakness with using the data-dependent proposal is that we need to simulate from a more complex distribution $p(x_t|x_{t-1}, y_t)$ and also need to work out the density $p(y_t|x_{t-1})$.

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