Introduction and Techniques
in Financial Mathematics
UiO-STK4510
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Solution Exam

Problem 1

a) A Brownian motion is a stochastic process W satisfying:

1. W has continuous paths P-a.s.,
2. $W_0 = 0, P$-a.s.,
3. W has independent increments,
4. for all $0 \leq s < t$, the law of $W_t - W_s$ is a $\mathcal{N}(0, (t-s))$.

A $\mathbb{F}$-Brownian motion is a process W which $\mathbb{F}$-adapted and satisfies:

1. W has continuous paths P-a.s.,
2. $W_0 = 0, P$-a.s.,
3. for all $0 \leq s < t$, the random variable $W_t - W_s$ is independent of $\mathcal{F}_s$,
4. for all $0 \leq s < t$, the law of $W_t - W_s$ is a $\mathcal{N}(0, (t-s))$.

b) An alternative definition of Brownian motion is that it is a Gaussian process with continuous paths and mean function identically equal to zero and covariance function equal to $\min(s,t)$. For $a > 0$, the process $X_t = a^{-1/2} W_{at}$ has continuous paths P-a.s. because W is a Brownian motion and it has continuos paths P-a.s.. Moreover, X_t is a Gaussian process because, for $0 \leq t_1 < t_2 < \cdots < t_n$ the vector $(a^{-1/2} W_{at_1}, ..., a^{-1/2} W_{at_n})$ is multivariate Gaussian (it is a linear transform of $(W_{at_1}, ..., W_{at_n})$, which is multivariate Gaussian because W is a Gaussian process). Finally, the mean and covariance function of X are given by

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}[a^{-1/2} W_{at}] = a^{-1/2} \mathbb{E}[W_{at}] = 0, \\ \mathrm{Cov}(X_s, X_t) &= \mathbb{E}[(X_t - \mathbb{E}[X_t])(X_s - \mathbb{E}[X_s])] = \mathbb{E}[X_s X_t] \\ &= \mathbb{E}[a^{-1/2} W_{as} a^{-1/2} W_{at}] = \frac{1}{a} \mathbb{E}[W_{as} W_{at}] \\ &= \frac{1}{a} \min(as, at) = \min(s,t), \end{aligned}$$

where we have used that W is a Brownian motion. If $a > 1$ we have that X_t cannot be a $\mathbb{F}$-Brownian motion because X_t depend on W_{at}, which is not $\mathcal{F}_t$ measurable and, hence, X is not $\mathbb{F}$-adapted. On the other hand, if $a < 1$ then the increments

$$X_t - X_s = a^{-1/2}(W_{at} - W_{as})$$

are not independent from from $\mathcal{F}_s$ for all $0 \leq s < t$. If we choose $s < t$, satisfying also that $a < s/t$, we get that $W_{at} - W_{as}$ is $\mathcal{F}_s$-measurable and, therefore, $X_t - X_s$ is also$\mathcal{F}_s$-measurable and cannot be independent of $\mathcal{F}_s$.

c) Assume that we have $n+1$ daily observation of the price process $\{S_i\}_{i=0,...,n}$. Let $r_i = \log(\frac{S_i}{S_{i-1}}), i = 1, ..., n$ be the series of logreturns. The maximum likelihood estimators of μ and σ (anualized) are given by

$$\hat{\mu} = \frac{1}{n\Delta t}\sum_{i=1}^{n} r_i,$$

and

$$\widehat{\sigma} = \sqrt{\frac{1}{(n-1)\Delta t}\sum_{i=1}^{n}(r_i - \hat{\mu})^2},$$

where $\Delta t = 1/252$.

d) The t arbitrage free price of a contingent claim H of the form $h(S_T)$ is given by $f(t, S_t)$, where the function $f(t, x)$ is given by the following expectation $e^{-r(T-t)}\mathbb{E}[h(Z_T^{t,x})]$ where $\log\left(Z_T^{t,x}\right) \sim \mathcal{N}(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t))$ The Monte Carlo method is based in the law of large numbers that roughly states that we can approximate an expectation by its sample mean. In our case, let $\{y_i\}_{i=1,...,N}$ be N independent outcomes of a $\mathcal{N}(0,1)$. Then

$$z_i = x\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}y_i\right), \qquad i = 1, ..., N,$$

are N independent outcomes of $Z_T^{t,x}$ and we approximate $f(t, x)$ by

$$e^{-r(T-t)}\sum_{i=1}^{n} h(z_i).$$

The central limit theorem yields a rate of convergence for the method of $1/\sqrt{N}$.

Problem 2

a) $L^2_{a,T}$ is the class of processes that are measurable, $\mathbb{F}$-adapted and square integrable (with respect to $\lambda \otimes P$). That is, the class of measurable processes h such that h_t is $\mathcal{F}_t$-measurable for all $t \in [0, T]$ and

$$\|h\|^2_{L^2_{a,T}} := \mathbb{E}\left[\int_0^T |h_t|^2 dt\right] < \infty.$$

The Itô isometry states that if $h \in L^2_{a,T}$ then

$$E\left[\left(\int_0^T h_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T |h_t|^2 dt\right] = \|h\|^2_{L^2_{a,T}}.$$

W belongs to $L^2_{a,T}$ because it is measurable, $\mathbb{F}$-adapted and

$$\mathbb{E}\left[\int_0^T |W_t|^2 dt\right] = \int_0^T \mathbb{E}\left[|W_t|^2\right] dt = \int_0^T t dt = \frac{T^2}{2} < \infty.$$

By the Itô isometry we have that

$$E\left[\left(\int_0^T W_t dW_t\right)^2\right] = \frac{T^2}{2}.$$

b) The Itô formula is as follows: Let $f \in C^{1,2}([0,T] \times \mathbb{R})$, that is, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are continuous functions. Assume that

$$\mathbb{E}\left[\int_0^T \left\{\left(\frac{\partial f}{\partial t}(t, W_t)\right)^2 + \left(\frac{\partial^2 f}{\partial x^2}(t, W_t)\right)^2 + \left(\frac{\partial f}{\partial x}(t, W_t)\right)^2\right\} dt\right] < \infty.$$

Then,

$$f(t, W_t) = f(0,0) + \int_0^t \left\{ \frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s) \right\} ds + \int_0^t \frac{\partial f}{\partial x}(s, W_s) dW_s,$$

for $0 \le t \le T$ and it is also an L^2-Itô process. We have that $e^{t/2}\sin(W_t)$ is of the form $f(t, W_t)$ with $f(t,x) = e^{t/2}\sin(x)$. Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} f(t,x) &= \frac{1}{2} e^{t/2}\sin(x) = \frac{1}{2} f(t,x), \\ \frac{\partial f}{\partial x}(t,x) &= e^{t/2}\cos(x) \\ \frac{\partial^2 f}{\partial x^2}(t,x) &= -e^{t/2}\sin(x) = -f(t,x). \end{aligned}$$

Hence, applying Itô's formula, we get that

$$\begin{aligned} e^{t/2}\sin(W_t) &= f(0,0) + \int_0^t \left\{ \frac{1}{2} f(s, W_s) - \frac{1}{2} f(s, W_s) \right\} ds + \int_0^t e^{s/2}\cos(W_s) dW_s \\ &= \int_0^t e^{s/2}\cos(W_s) dW_s. \end{aligned}$$

c) The martingale representation theorem is as follows: Let $M = \{M_t\}_{t\in[0,T]}$ be a square integrable $\mathbb{F}$-martingale. Then, there exists a unique $h \in L^2_{a,T}$ such that

$$M_t = \mathbb{E}[M_0] + \int_0^t h_s dW_s.$$

As

$$e^{t/2}\sin(W_t) = \int_0^t e^{s/2}\cos(W_s) dW_s,$$

and

$$\mathbb{E}[\int_0^T \left| e^{s/2}\cos(W_s) \right|^2 ds] \le Te^T < \infty,$$

we have that $h_s = e^{s/2}\cos(W_s)$ is the kernel in the martingale representation of $e^{t/2}\sin(W_t)$.

d) A sufficient condition is that $f(t,x)$ satisfies the partial differential equation

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0$$

and

$$\mathbb{E}\left[\int_0^T \left(\frac{\partial f}{\partial x}(t, W_t) \right)^2 dt \right] < \infty.$$

Problem 3

a) A portfolio $\phi = \{\phi^0, \phi^1\}$ is self-financing if its value process

$$V_t(\phi) = \phi_t^0 e^{rt} + \phi_t^1 S_t \tag{1}$$

is an Itô process satisfying

$$dV_t(\phi) = re^{rt}\phi_t^0 dt + \phi_t^1 dS_t. \tag{2}$$

By the integration by parts formula we have that the discounted value of the portfolio has the dynamics

$$d\tilde{V}_t(\phi) = d(e^{-rt}V_t(\phi)) = -re^{-rt}V_t(\phi)dt + e^{-rt}dV_t(\phi) + (de^{-rt})(dS_t),$$

plugging equations (1) and (2) taking into account that $(de^{-rt})(dS_t) = 0$, we get that

$$\begin{aligned} d\tilde{V}_t(\phi) &= -re^{-rt}\phi_t^0 e^{rt}dt - re^{-rt}\phi_t^1 S_t dt + e^{-rt} r e^{rt}\phi_t^0 dt + e^{-rt}\phi_t^1 dS_t \\ &= -re^{-rt}\phi_t^1 S_t dt + e^{-rt}\phi_t^1 dS_t = \phi_t^1 \left\{ d(e^{-rt})S_t + e^{-rt}dS_t \right\} \\ &= \phi_t^1 d(e^{-rt}S_t) = \phi_t^1 d(\tilde{S}_t), \end{aligned}$$

which is the resu

b) Consider the measure Q on $\mathcal{F}_T$ given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T \right).$$

By Girsanov's Theorem we have that

$$\tilde{W}_t = \frac{\mu - r}{\sigma} t + W_t$$

is a Brownian motion under Q. We can write the dynamics of S_t using $\tilde{W}$, i.e,

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t = \mu S_t dt + \sigma S_t \{ d\tilde{W}_t - \frac{\mu - r}{\sigma} dt \} \\ &= \left\{ \mu - \sigma \frac{\mu - r}{\sigma} \right\} S_t dt + \sigma S_t d\tilde{W}_t \\ &= r S_t dt + \sigma S_t d\tilde{W}_t, \end{aligned}$$

and using the integration by parts formula, taking into account that $d(e^{-rt})dS_t = 0$ we get that

$$\begin{aligned} d(\tilde{S}_t) &= d(e^{-rt}S_t) = d(e^{-rt})S_t + e^{-rt}dS_t + d(e^{-rt})dS_t \\ &= -re^{-rt}S_t dt + e^{-rt}dS_t = -re^{-rt}S_t dt + e^{-rt}\{ r S_t dt + \sigma S_t d\tilde{W}_t \} \\ &= \sigma e^{-rt} S_t d\tilde{W}_t = \sigma \tilde{S}_t d\tilde{W}_t, \end{aligned} \tag{3}$$

which can be written explicitly as

$$\tilde{S}_t = \exp\left(\sigma \tilde{W}_t - \frac{\sigma^2}{2} t \right).$$

As $\tilde{W}$ is a Brownian motion under Q, we get that $\tilde{S}_t$ is an exponential martingale under Q. Alternatively, equation (3) says that $\tilde{S}_t$ is an Itô integral with respect to a Brownian motion under Q and, hence, a martingale under Q.

c) Let ϕ be an admissible replicating portfolio for H. The discounted value process of ϕ is a martingale under Q because ϕ is admissible and, in addition, $\tilde{V}_T(\phi) = e^{-rT}H$. Hence,

$$e^{-rt}V_t(\phi) = \tilde{V}_t(\phi) = \mathbb{E}_Q\left[\tilde{V}_T(\phi) | \mathcal{F}_t \right] = \mathbb{E}_Q\left[e^{-rT} H | \mathcal{F}_t \right],$$

which yields $V_t(\phi) = \mathbb{E}_Q\left[e^{-r(T-t)} H | \mathcal{F}_t \right]$ and as $\pi_t(\phi) = V_t(\phi)$ we get the result.

d) In the Black-Scholes model we have the following formulas for the price process and hedging strategy of a contingent claim of the form $H = h(S_T)$.

$$\pi_t(H) = f(t, S_t),$$

and

$$(\phi_t^0, \phi_t^1) = \left(e^{-rt}\left\{ f(t, S_t) - S_t \frac{\partial f}{\partial x}(t, S_t) \right\}, \frac{\partial f}{\partial x}(t, S_t) \right), \qquad t \in [0, T],$$

where

$$f(t, x) = e^{-r(T-t)}\mathbb{E}[h(Z_T^{t,x})],$$

and $\log Z_T^{t,x} \sim \mathcal{N}\left(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$. In our case $h(x) = x^2$ and, using the moment generating function of a normal distribution, we get that

$$\begin{aligned} f(t,x) &= e^{-r(T-t)}\mathbb{E}[(Z_T^{t,x})^2] = e^{-r(T-t)}\mathbb{E}[(\exp(\log(Z_T^{t,x})))^2] \\ &= e^{-r(T-t)}\mathbb{E}[\exp(2\log(Z_T^{t,x}))] = e^{-r(T-t)}\mathbb{E}[\exp(2\log(Z_T^{t,x}))] \\ &= e^{-r(T-t)}\exp\left(2\left(\log(x) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right) + \frac{\sigma^2(T-t)}{2}4\right) \\ &= x^2\exp\left(r(T-t) + \sigma^2(T-t)\right) \end{aligned}$$

and $\frac{\partial f}{\partial x}(t,x) = 2x\exp\left(r(T-t) + \sigma^2(T-t)\right)$. Plugging $x = S_t$ in $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ we get the results for $\pi_t(H)$ and (ϕ_t^0, ϕ_t^1).

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

a) X is a random variable if the mapping X is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. For any $B \in \mathcal{B}(\mathbb{R})$ we have that

$$X^{-1}(B) = \begin{cases} \Omega & \text{if} \quad K \in B \\ \varnothing & \text{if} \quad K \notin B \end{cases}.$$

As $\varnothing$ and Ω belong to any σ-algebra $\mathcal{F}$, one has that constant functions are always random variables.

b) Q is absolutely continuous with respect to P ($Q \ll P$) if

$$\forall B \in \mathcal{F}, \qquad P(A) = 0 \Rightarrow Q(A) = 0.$$

P and Q are equivalent if $Q \ll P$ and $P \ll Q$.

c) Let P and Q two probability measures on a measurable space $(\Omega, \mathcal{F})$. Then the following two statements are equivalent:

1. $Q \ll P$.
2. There exists a measurable function $f : \Omega \to \mathbb{R}_+$ such that $f = \frac{dQ}{dP}$.

If $Q \ll P$ and $g : \Omega \to \mathbb{R}$ is a measurable function we have that

$$\mathbb{E}_Q[g] = \mathbb{E}_P[g\frac{dQ}{dP}]. \tag{4}$$

Therefore, if $R \ll Q \ll P$, for any $B \in \mathcal{F}$, we have that

$$R(B) = \mathbb{E}_R[\mathbf{1}_B] = \mathbb{E}_Q[\mathbf{1}_B\frac{dR}{dQ}] = \mathbb{E}_P[\mathbf{1}_B\frac{dR}{dQ}\frac{dQ}{dP}], \tag{5}$$

where we have used twice the property (4). Equality (5) yields the result.

d) $\mathbb{E}[X|\mathcal{G}]$ is the unique $\mathcal{G}$-measurable random variable satisfying

$$\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]1_B], \qquad B \in \mathcal{G}.$$

Let's check that $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$. $\mathbb{E}[X]$ is constant and by *a*) is measureable with respect to any σ-algebra, in particular $\mathcal{G}$. Moreover, for any $B \in \mathcal{G}$, one has

$$\mathbb{E}[\mathbb{E}[X]\mathbf{1}_B] = \mathbb{E}[X]P(B).$$

On the other hand using that X and $\mathbf{1}_B$ are independent we get that

$$\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[X]\mathbb{E}[\mathbf{1}_B] = \mathbb{E}[X]P(B),$$

so we can conclude.