UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4510 — Introduction to methods and techniques in financial mathematics.

Day of examination: Wednesday, December 5, 2012.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All exercises count equally in the evaluation.

Exercise 1

a) What is a martingale?

b) Define a Brownian motion $B(t)$ and its filtration $\mathcal{F}_t$

c) Show that $B(t)$ is a martingale (with respect to its filtration $\mathcal{F}_t$). Analyse this based on the definition of a martingale.

d) For two constants a and b different than zero, when is $M(t) = aB^2(t) - bt$ a martingale (with respect to $\mathcal{F}_t$)? Analyse this based on the definition of a martingale.

e) State the martingale representation theorem.

f) Can the martingale representation theorem be applied on the processes $B(t)$ and $B^2(t) - t$? If yes, find the explicit representation.

Exercise 2

Let a stock price $S(t)$ be defined as a geometric Brownian motion with dynamics

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dB(t)$$

where B is a Brownian motion and α, σ are two constant (the latter naturally positive).

a) Find $S(t)$.

b) Show by Girsanov's Theorem that there exists a probability Q such $\exp(-rt)S(t)$ becomes a martingale for this probability. State the dynamics of S under Q. Here, r is a positive constant being the risk-free interest rate.

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c) At a time t you are offered to buy an option which pays the difference between the stock price at two different times in the future, let us say times T_1 and T_2 with $T_1 < T_2$. The option pays $S(T_2) - S(T_1)$ at time T_2 whenever this is positive, and zero otherwise. Compute the arbitrage-free price of this option at time $t \leq T_1$.

d) Suppose you know how to hedge (replicate) this option using the risk-free asset and the stock. Explain how you can arbitrage on a market price for the option being *bigger* than the price you found in c).

Exercise 3

Let $B(t)$ be a Brownian motion.

a) Use Itô's Formula to show that

$$\int_0^1 s\,dB(s) = B(1) - \int_0^1 B(s)\,ds$$

b) Use *the definition* of Itô integration to compute $\int_0^1 s\,dB(s)$.

END