UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4510 — Introduction to methods and techniques in financial mathematics

Day of examination: Thursday, December 5, 2013

Examination hours: 09.00 – 13.00

This exercise set consists of 4 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the exercise set is complete before you attempt to answer anything.

All exercises count equally

Exercise 1

Let $B(t)$ for $t \geq 0$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $\mathcal{F}_t$ is the filtration.

1a

What is the probability distribution of $B(t)$ for a given time $t > 0$?

1b

Calculate

$$\mathbb{E}[(B(t) - B(s))(B(v) - B(u))]$$

for $t > s \geq v > u$. Explain how you found your answer.

1c

Let $S(t)$ be the stock price at time t defined as a geometric Brownian motion of the form

$$S(t) = S(0)\exp(\mu t + \sigma B(t))$$

for μ and $\sigma > 0$ being constants. What statistical properties do the logreturns (logarithmic returns) of the stock price have?

1d

Explain why the process

$$Y(t) = \int_0^t X(s)\,dB(s)$$

is $\mathcal{F}_t$-adapted, where X is an Ito integrable process.

(Continued on page 2.)

Exercise 2

2a

Let $X(t)$ be an Ito diffusion, and $f : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ a function which is continuously differentiable with respect to time t and twice continuously differentiable with respect to space x. State the Ito Formula for the process $Y(t) = f(t, X(t))$.

2b

$X(t)$ is a geometric Brownian motion

$$X(t) = \exp(t + 2dB(t))$$

where B is a Brownian motion. What is $dX(t)$?

2c

Use Ito's Formula to find $Y(t)$ where $Y(t)$ is the solution of the stochastic differential equation

$$dY(t) = -\alpha Y(t)\,dt + \sigma dB(t)$$

where α, σ are two positive constants and $Y(0) = y$, is a given constant.

Exercise 3

3a

Define what is a martingale $M(t)$ with respect to the filtration $\mathcal{F}_t$ generated by Brownian motion $B(t)$. Show that $\mathbb{E}[M(t)] = M(0)$, for $M(0)$ a constant.

3b

We know from the martingale representation theorem that if $M(t)$ has finite variance, there exists an Ito integrable process $Y(t)$ such that

$$M(t) = M(0) + \int_0^t Y(s)\,dB(s)$$

Show that this representation is unique.

3c

Apply Ito's Formula to find a process $Z(t)$ such that

$$M(t) = B^3(t) - Z(t)$$

is a martingale. What is the process Y in the martingale representation theorem for this martingale?

(Continued on page 3.)

3d

Let $S(t)$ be a geometric Brownian motion

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t)$$

Find a probability Q and a Q-Brownian motion $W(t)$ such that the process $e^{-rt}S(t)$ becomes a Q-martingale. Here $r > 0$ is the risk-free interest rate.

3e

Let X be a contingent claim, that is, an $\mathcal{F}_T$-measurable random variable with finite variance (with respect to Q). Define what we mean by a replicating portfolio of X, and show that the discounted value of this replicating portfolio is a Q-martingale.

3f

Show that the arbitrage-free price of X is

$$P(t) = e^{-r(T-t)}\mathbb{E}_Q[X \mid \mathcal{F}_t]$$

for $t \leq T$.

Exercise 4

In this Exercise you are going to derive an approximation of the price of a call option written on a portfolio of two assets. Let the price of the two assets be given by a bivariate geometric Brownian motion

$$\begin{aligned} dS_1(t) &= rS_1(t)\,dt + S_1(t)\sigma_1\,dW_1(t) \\ dS_2(t) &= rS_2(t)\,dt + S_2(t)\sigma_2\left(\rho\,dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right), \end{aligned}$$

under the equivalent martingale measure Q.

4a

Show that

$$\begin{aligned} S_1(t) &= S_1(0)\exp\left(\left(r - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 W_1(t)\right) \\ S_2(t) &= S_2(0)\exp\left(\left(r - \frac{1}{2}\sigma_2^2\right)t + \sigma_2\left(\rho W_1(t) + \sqrt{1-\rho^2}W_2(t)\right)\right) \end{aligned}$$

4b

We want to price a call option written on a portfolio consisting of the sum of asset 1 and asset 2. The exercise time is T and strike price is K. Argue that the arbitrage-free price of this option at time $t \leq T$ is given by

$$P(t) = e^{-r(T-t)}\mathbb{E}_Q\left[\max\left(S_1(T) + S_2(T) - K, 0\right) \mid \mathcal{F}_t\right]$$

where $r > 0$ is the risk-free interest rate and $\mathcal{F}_t$ the given filtration.

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4c

There is no analytic pricing formula for this option, so we will consider an approximation in this Exercise for $P(0)$, that is, the price at time $t = 0$. Introduce a random variable X which is normally distributed under Q with mean a and variance b^2, for two constants a and $b > 0$. Derive the equations for a and b such that

$$\mathbb{E}_Q[S_1(T) + S_2(T)] = \mathbb{E}_Q[\exp(X)]$$
$$\mathbb{E}_Q[(S_1(T) + S_2(T))^2] = \mathbb{E}_Q[\exp(2X)]$$

Note: you do not need to solve the equations!

4d

We approximate the price $P(0)$ by

$$P(0) \approx e^{-rT}\mathbb{E}_Q\left[\max\left(e^X - K, 0\right)\right]$$

Compute the approximate price.

4e

Explain how you can modify the above analysis to approximate $P(t)$ for an arbitrary time $t \leq T$.

END