Introduction to methods and techniques in financial mathematics (STK 4510) Solutions to the exam, 04.12.2014

Problem 1 Denote by $X(i) := \log(S(t_{i+1})/S(t_i))$ the $i-$th log-return. Then we use the MLE estimators $\widehat{\mu} = \frac{1}{N}\sum_{i=1}^{N} X(i)$ and $\widehat{\sigma} = \frac{1}{N}\sum_{i=1}^{N}(X(i) - \widehat{\mu})^2$ for $N = 5$ to obtain

$$\widehat{\mu} = 0.00148286, \widehat{\sigma} = 0.00189397$$

Problem 2 We have to study the solutions $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ to the linear system of equations (see e.g. Th. 5.20 of the course manuscript):

$$\begin{aligned} \alpha_1 - r + \sigma_{11}\lambda_1 + \sigma_{12}\lambda_2 &= 0 \\ \alpha_2 - r + \sigma_{21}\lambda_1 + \sigma_{22}\lambda_2 &= 0. \end{aligned}$$

Denote by $\sigma = (\sigma_{ij})_{1\le i,j\le 2}$. Then $\det(\sigma) = 0.30 \cdot 0.25 - 0.20 \cdot 0.375 = 0$. So $rank(\sigma) < m = 2$. The latter implies (by Th. 5.20) that the BS-market is not complete.

Substitution of λ_1 in the first equation gives

$$\alpha_1 - r + \frac{\sigma_{11}}{\sigma_{21}}(r - \alpha_2) - \frac{\det(\sigma)}{\sigma_{21}}\lambda_2 = \alpha_1 - r + \frac{\sigma_{11}}{\sigma_{21}}(r - \alpha_2) = 0. \qquad (1)$$

So the market has no arbitrage if and only if equation (1) holds.

Problem 3 (i) Let $\mathcal{F}_t$, $0 \le t \le T$ be a filtration. A process $X(t)$ is a $\mathcal{F}_t$-martingale iff (i) $E[|X_t|] < \infty$ for all t (ii) X_t is $\mathcal{F}_t$-adapted and (iii) $E[X_t | \mathcal{F}_s] = X_s$ for $t \ge s$.

(ii) Conditions (i) and (ii) are clearly fulfilled by B_t. Since $B_t - B_s$ is independent of B_s for $t > s$ we find

$$\begin{aligned} E[B_t | \mathcal{F}_s] &= E[B_t - B_s + B_s | \mathcal{F}_s] = E[B_t - B_s | \mathcal{F}_s] + E[B_s | \mathcal{F}_s] \\ &= E[B_t - B_s] + B_s = B_s. \end{aligned}$$

As for $(B_t)^2$ we know by Itô's Lemma that $(B_t)^2 = 2\int_0^t B_s dB_s + t$. Since $E[\int_0^T (B_s)^2 ds] < \infty$, $2\int_0^{\cdot} B_s dB_s$ is a martingale. Hence we get

$$E[2\int_0^t B_s dB_s + t \,|\mathcal{F}_s] = \int_0^s B_s dB_s + t \neq \int_0^s B_s dB_s + s \text{ for } t > s.$$

So the second process is not a martingale.

(iii) Observe that $\frac{(x+h)^3}{3} - \frac{x^3}{3} = x^2h + xh^2 + \frac{1}{3}h^3$. From this we get For $0 = t_0 < t_1 < ... < t_n = t$ that

$$\begin{aligned} \frac{B_t^3}{3} &= \sum_{i=1}^{n} \frac{(B_{t_{i-1}} + (B_{t_i} - B_{t_{i-1}}))^3}{3} - \frac{B_{t_{i-1}}^3}{3} \\ &= \sum_{i=1}^{n}(B_{t_{i-1}})^2(B_{t_i} - B_{t_{i-1}}) + \sum_{i=1}^{n} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})^2 + \frac{1}{3}\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^3. \end{aligned}$$

If the mesh of the partition tends to zero the first sum on the right hand side goes by the definition of stochastic integrals to $\int_0^t B_s^2 dB_s$ (for a subsequence). The second sum converges by the independent and stationary increments of B_t to $\int_0^t B_s ds$ (for a subsequence). The last sum tends to zero (for a subsequence), since $E[\left|B_{t_i} - B_{t_{i-1}}\right|^3] \leq 6\left|t_i - t_{i-1}\right|^{3/2}$ (which is obtained by means of the Gaussian density of $B_{t_i} - B_{t_{i-1}}$).

Problem 4 (i) Payoff $X = 1_{[20,30]}(S(T))$. We know that

$$ClaimValue_t = E_{\widetilde{P}}[e^{-r(T-t)}X \,|\mathcal{F}_t]\,,$$

where $\widetilde{P}$ is a probabilty measure such that $\widetilde{S}(t) := e^{-rt}S(t)$ is a martingale. By Itô's Lemma we have

$$\widetilde{S}(t) = x + \int_0^t (\mu - r)\widetilde{S}(s)ds + \int_0^t \sigma\widetilde{S}(s)dB_s.$$

Defining $\widetilde{B}_t := B_t - \int_0^t \lambda ds$, where $\lambda = \frac{r-\mu}{\sigma}$ we know by Girsanov's theorem that $\widetilde{B}_t$ is a Brownian motion w.r.t. $\widetilde{P}$ given by

$$\begin{aligned} \widetilde{P}(A) &:= E[1_A Z_T], A \in \mathcal{F}, \\ Z_t &:= \exp(\lambda B_t - \frac{1}{2}\lambda^2 t), 0 \leq t \leq T. \end{aligned}$$

Then

$$\begin{aligned} \widetilde{S}(t) &= x + \int_0^t (\mu - r)S(s)ds + \int_0^t \sigma S(s)d(\widetilde{B}_s + \lambda s) \\ &= \int_0^t ((\mu - r) + \lambda\sigma)S(s)ds + \int_0^t \sigma S(s)d\widetilde{B}_s = \int_0^t \sigma S(s)d\widetilde{B}_s, \end{aligned}$$

So $\widetilde{S}(t)$ is a martingale under $\widetilde{P}$. From the Itô-Lemma we know that

$$S(t) = x\exp((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t) = x\exp((r - \frac{1}{2}\sigma^2)t + \sigma\widetilde{B}_t).$$

So for $t = T$ we get

$$\begin{aligned} S(T) &= x\exp((r - \frac{1}{2}\sigma^2)t + \sigma\widetilde{B}_t)\exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{B}_T - \widetilde{B}_t)) \\ &= S(t)\exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{B}_T - \widetilde{B}_t)). \end{aligned}$$

$S(t)$ is $\mathcal{F}_t$-adapted and as a function of $\widetilde{B}_t$ independent of $(\widetilde{B}_T - \widetilde{B}_t)$. Further $(\widetilde{B}_T - \widetilde{B}_t)$ is independent of the events in $\mathcal{F}_t$. So we can treat $S(t)$ as a constant in the above conditional expectation and drop the conditioning on $\mathcal{F}_t$. Thus we get

$$ClaimValue_t = C(t, S(t)),$$

where

$$\begin{aligned} C(t,y) &:= e^{-r(T-t)}E_{\widetilde{P}}[1_{[20,30]}(y \cdot \exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{B}_T - \widetilde{B}_t)))] \\ &= e^{-r(T-t)}\widetilde{P}(10 \le y \cdot \exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{B}_T - \widetilde{B}_t)) \le 20) \\ &= e^{-r(T-t)}\widetilde{P}((d_1 \le \xi \le d_2) = e^{-r(T-t)}(\Phi(d_2) - \Phi(d_1)), \end{aligned} \tag{2}$$

where $\xi \sim \mathcal{N}(0,1)$, Φ the standard normal distribution function and

$$\begin{aligned} d_1 &:= (\log(20/y) - (r - \frac{1}{2}\sigma^2)(T-t))/\sigma\sqrt{T-t}, \\ d_2 &:= (\log(30/y) - (r - \frac{1}{2}\sigma^2)(T-t))/\sigma\sqrt{T-t}. \end{aligned}$$

(ii) Differentiation of the right hand side of (2) w.r.t. y in connection with the chain rule gives

$$\frac{\partial}{\partial y}C(t,y) = -e^{-r(T-t)}\frac{1}{y\sigma\sqrt{T-t}}(\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}d_2^2) - \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}d_1^2)).$$

Hence the replicating stock strategy at time t is given by $\frac{\partial}{\partial y}C(t,y)\Big|_{y=S(t)}$.

Problem 5 Denote by C_i the solution of the Black-Scholes partial differential equation w.r.t. the terminal condition $C_i(T,x) = \max(0, x-K)$ and volatility $\sigma_i, i = 1,2$.

Using Girsanov's theorem, we find that

$$dS(t) = rS(t)dt + \widetilde{\sigma}(t)S(t)d\widetilde{B}_t,$$

where

$$\widetilde{B}_t = B_t - \int_0^t \frac{r-\mu}{\widetilde{\sigma}(s)} ds$$

is a Brownian motion under the probability measure $\widetilde{P}$ given by

$$\widetilde{P}(A) = E[\mathbf{1}_A Z_T], A \in \mathcal{F}$$

for

$$Z_t = \exp\Big(\int_0^t \frac{r-\mu}{\widetilde{\sigma}(s)} dB_s - \frac{1}{2}\int_0^t \Big(\frac{r-\mu}{\widetilde{\sigma}(s)}\Big)^2 ds\Big), 0 \le t \le T.$$

Then Itô's Lemma applied to $S(t), 0 \le t \le T$ and $e^{-rt}C_1(t,x)$ under $\widetilde{P}$ gives

$$\begin{aligned}
& e^{-rt}C(t,S(t)) \\
= \; & C_1(0,S(0)) + \int_0^t \{(-r)e^{-rs}C_1(s,S(s)) + e^{-rs}\frac{\partial C_1}{\partial s}(s,S(s))\}ds \\
& + \int_0^t e^{-rs}\frac{\partial C_1}{\partial x}(s,S(s))rS(s)ds + \int_0^t e^{-rs}\frac{\partial C_1}{\partial x}(s,S(s))\widetilde{\sigma}(s)S(s)d\widetilde{B}_s \\
& + \frac{1}{2}\int_0^t e^{-rs}\frac{\partial^2 C_1}{\partial x^2}(s,S(s))(\widetilde{\sigma}(s)S(s))^2 ds. \qquad (3)
\end{aligned}$$

We also obtain from differentiation of the stochastic representation of C_1 (see exercises) that

$$\frac{\partial C_1}{\partial x}(t,x) = \Phi(d_1), \qquad (4)$$

where Φ is the standard normal distribution function and

$$d_1 = (\log(x/K) + (r + \frac{\sigma_1^2}{2})(T-t))/\sigma_1\sqrt{T-t}.$$

So $\frac{\partial C_1}{\partial x}$ is bounded. Further, we also see that $E_{\widetilde{P}}[(\widetilde{S}(t))^2] \le (S(0))^2 \exp(\sigma_2^2 t)$. Hence

$$E_{\widetilde{P}}\Big[\int_0^T \Big(e^{-rs}\frac{\partial C_1}{\partial x}(s,S(s))\widetilde{\sigma}(s)S(s)\Big)^2 ds\Big] < \infty.$$

Thus $M_t := \int_0^t e^{-rs}\frac{\partial C_1}{\partial x}(s,S(s))\widetilde{\sigma}(s)S(s)d\widetilde{B}_s$ is a $\widetilde{P}-$martingale. On the other hand, we see by differentiating the right hand side of (4) that

$$\frac{\partial^2 C_1}{\partial x^2}(t,x) \geq 0$$

So taking expectation on both sides of (3) in connection with the Black–Scholes-PDE gives

$$\begin{aligned}
& E_{\widetilde{P}}[e^{-rt}C(t,S(t))] \\
= \; & C_1(0,S(0)) + E_{\widetilde{P}}[\int_0^t \{(-r)e^{-rs}C_1(s,S(s)) + e^{-rs}\frac{\partial C_1}{\partial s}(s,S(s))\}ds \\
& + \int_0^t e^{-rs}\frac{\partial C_1}{\partial x}(s,S(s))rS(s)ds + \frac{1}{2}\int_0^t e^{-rs}\frac{\partial^2 C_1}{\partial x^2}(s,S(s))(\widetilde{\sigma}(s)S(s))^2 ds] \\
\geq \; & C_1(0,S(0)) + E_{\widetilde{P}}[\int_0^t \{(-r)e^{-rs}C_1(s,S(s)) + e^{-rs}\frac{\partial C_1}{\partial s}(s,S(s))\}ds \\
& + \int_0^t e^{-rs}\frac{\partial C_1}{\partial x}(s,S(s))rS(s)ds + \frac{1}{2}\int_0^t e^{-rs}\frac{\partial^2 C_1}{\partial x^2}(s,S(s))(\sigma_1 S(s))^2 ds] \\
= \; & C_1(0,S(0)) \\
= \; & p_{1.}
\end{aligned}$$

We have that $E_{\widetilde{P}}[e^{-rT}C(T,S(T))] = p$. So $p \geq p_1$. Similarly, we get $p \leq p_2$.