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Compulsory problem set 2 – solutions
ECON3120/4120 Mathematics 2, autumn 2003

Problem 1

(a) The determinant of the coefficient matrix is

$$\begin{vmatrix} 3 & -2 & 4 \\ 1 & 2 & -4 \\ -1 & 1 & p \end{vmatrix} = 3\begin{vmatrix} 2 & -4 \\ 1 & p \end{vmatrix} - (-2)\begin{vmatrix} 1 & -4 \\ -1 & p \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= 3(2p+4) + 2(p-4) + 4 \cdot 3 = 8p + 16.$$

By Cramer's rule the equation system has a unique solution if and only if the determinant is different from 0, that is if and only if $p \neq -2$.

To see what happens if $p = -2$, we can use elementary operations on the system:

$$\begin{pmatrix} 3 & -2 & 4 & 5 \\ 1 & 2 & -4 & a \\ -1 & 1 & -2 & 1 \end{pmatrix} \begin{matrix} \leftarrow \\ -3 \quad 1 \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 0 & -8 & 16 & 5-3a \\ 1 & 2 & -4 & a \\ 0 & 3 & -6 & a+1 \end{pmatrix} \cdot (-\tfrac{1}{8})$$

$$\sim \begin{pmatrix} 0 & 1 & -2 & \frac{1}{8}(3a-5) \\ 1 & 2 & -4 & a \\ 0 & 3 & -6 & a+1 \end{pmatrix} \begin{matrix} -2 \; -3 \\ \leftarrow \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 0 & 1 & -2 & \frac{1}{8}(3a-5) \\ 1 & 0 & 0 & \frac{1}{4}(a+5) \\ 0 & 0 & 0 & \frac{1}{8}(23-a) \end{pmatrix}$$

The last row in the last matrix represents the equation $0 = \frac{1}{8}(23 - a)$. Thus, the system has no solution if $a \neq 23$. On the other hand, if $a = 23$ then it obviously has infinitely many solutions.

Hence, the original system has no solution if $p = -2$ and $a \neq 23$ and infinitely many solutions if $p = -2$ and $a = 23$.

(b) Most of the work has already been done in part (a). With $a = 23$ and $p = -2$ we get the equation system

$$\left.\begin{aligned} x_2 - 2x_3 &= 8 \\ x_1 \qquad\qquad &= 7 \\ 0 &= 0 \end{aligned}\right\} \iff \left\{\begin{aligned} x_1 &= 7 \\ x_2 &= 2x_3 + 8 \end{aligned}\right.$$

The solution of this system can be written as $x_1 = 7$, $x_2 = 2t + 8$, $x_3 = t$, with t an arbitrary number. Thus, we have solutions with one degree of freedom.

Problem 2

(a) $|\mathbf{A}_a| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & a-1 & 1 \\ 1 & 2 & a+1 \end{vmatrix} \begin{matrix} -1 \\ \\ \leftarrow \end{matrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & a-1 & 1 \\ 0 & 0 & a-2 \end{vmatrix} = (a-1)(a-2).$

$\mathbf{A}_a$ has an inverse $\iff$ $a \neq 1$ and $a \neq 2$.

With $a = 3$ we get $\mathbf{A}_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ with $|\mathbf{A}_3| = 2 \cdot 1 = 2$.

The cofactors of the elements in $\mathbf{A}_3$ are

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} = 6, \quad C_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} = 1, \quad C_{13} = \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = -2$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = -2, \quad C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1, \quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4, \quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1, \quad C_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$$

Hence,

$$\mathbf{A}_3^{-1} = \frac{1}{|\mathbf{A}_3|} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 6 & -2 & -4 \\ 1 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -2 \\ 1/2 & 1/2 & -1/2 \\ -1 & 0 & 1 \end{pmatrix}$$

(It is a good idea to check that $\mathbf{A}_3\mathbf{A}_3^{-1} = \mathbf{I}\,$!)

(b) Since $|\mathbf{B}| = -4 \neq 0$, the matrix $\mathbf{B}$ has an inverse, $\mathbf{B}^{-1}$. Multiplication of $\mathbf{BX} = \mathbf{B}^2 + 2\mathbf{B}$ from the left with $\mathbf{B}^{-1}$ gives

$$\mathbf{X} = \mathbf{B}^{-1}(\mathbf{B}^2 + 2\mathbf{B}) = \mathbf{B} + 2\mathbf{I} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 2 & 2 \end{pmatrix}.$$

Comment: Remember that you were asked to find *all* solutions. It is immediately clear from the equation $\mathbf{BX} = \mathbf{B}^2 + 2\mathbf{B} = \mathbf{B}(\mathbf{B} + 2\mathbf{I})$ that $\mathbf{X} = \mathbf{B} + 2\mathbf{I}$ is a solution. The problem is to show that this is the *only* solution, and that is where we need the inverse of $\mathbf{B}$.

Problem 3

(a) With $f(x,y) = \dfrac{y-x}{x^2+y^2+1}$, we get

$$f_1'(x,y) = \frac{(-1)(x^2+y^2+1)-(y-x)2x}{(x^2+y^2+1)^2} = \frac{x^2-y^2-2xy-1}{(x^2+y^2+1)^2},$$

$$f_2'(x,y) = \frac{(x^2+y^2+1)-(y-x)2y}{(x^2+y^2+1)^2} = \frac{x^2-y^2+2xy+1}{(x^2+y^2+1)^2}.$$

(b) The stationary points of f are the solutions of the equations

$$x^2-y^2-2xy-1=0 \iff x^2-y^2 = 2xy+1 \qquad (1)$$
$$x^2-y^2+2xy+1=0 \iff x^2-y^2 = -2xy-1 \qquad (2)$$

It follows that $2xy+1=-2xy-1$, that is, $2xy=-1$. Inserting this in (1) or (2), we get $x^2=y^2$, so $y=\pm x$. Since $2xy$ is negative, x and y must have opposite signs, and therefore $y=-x$. Hence, $2x^2=-2xy=1$, and $x=\pm\frac{1}{2}\sqrt{2}$. Thus, the stationary points are

$$\left(\tfrac{1}{2}\sqrt{2}, -\tfrac{1}{2}\sqrt{2}\right) \quad \text{and} \quad \left(-\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}\right).$$

(c) The equation of the level curve, $\dfrac{y-x}{x^2+y^2+1} = c$, can be rewritten as

$$x^2+y^2+1 = \frac{1}{c}(y-x) \iff x^2+\frac{1}{c}x+y^2-\frac{1}{c}y+1=0.$$

Completing the squares, we get

$$\left(x+\frac{1}{2c}\right)^2 - \frac{1}{4c^2} + \left(y-\frac{1}{2c}\right)^2 - \frac{1}{4c^2} + 1 = 0,$$

that is,

$$\left(x+\frac{1}{2c}\right)^2 + \left(y-\frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1 = \frac{\frac{1}{2}-c^2}{c^2}.$$

Hence, if $0<|c|<\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$, the level curve $f(x,y)=c$ will be a circle centered at $\left(-\dfrac{1}{2c}, \dfrac{1}{2c}\right)$ and with radius $\sqrt{\dfrac{1}{2c^2}-1}$.

Notes on grading: I have indicated an approximate (and unofficial!) grade on each of the papers that I have corrected. This time I gave points on a scale from 0 to 50, and assigned grades as follows:

46–50: A, 39–45: B, 33–38: C, 26–32: D, 20–25: E.

On each paper I have noted both the letter grade and the point score. There were 42 papers in all, and the distribution of grades turned out as follows:

A: 5, B: 11, C: 15, D: 8, E: 3.

The highest score attained was 49 and the lowest was 20. If you got less than 35 points, then you should take that as a serious warning signal.

Remember that the grades given for this paper will *not* count towards your final grade for this course.

Concerning the exam: Remember that the material on differential equations in the handouts of chapter 5 from FMEA (or chapter 1 in MA II) will not be required on the exam in ECON3120/4120 this year.

The exam is an "open book exam", that is, you are allowed to use any printed or hand-written material, books, lecture notes, etc. Pocket calculators are also permitted.

Good luck!

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