Microeconomics 3200/4200:
Part 1

P. Piacquadio

p.g.piacquadio@econ.uio.no

September 14, 2017

Outline

Inputs and Outputs

- Firms are the economic actors that produce and supply commodities to the market.
- The **technology** of a firm can then be defined as the set of production processes that a firm can perform.
- A production process is an (instantaneous) transformation of **inputs**–commodities that are consumed by production–into **outputs**–commodities that result from production.

Inputs and Outputs

- Firms are the economic actors that produce and supply commodities to the market.
- The **technology** of a firm can then be defined as the set of production processes that a firm can perform.
- A production process is an (instantaneous) transformation of **inputs**–commodities that are consumed by production–into **outputs**–commodities that result from production.

Examples 1

- What are the combinations of inputs and outputs that are feasible?
- Given a vector of inputs, what is the largest amoung of outputs the firm can produce?

- With 1 input and 1 output, a typical production function looks like:

$$y \leq f(x),$$

where y is output, x is input, and f is the **production function**.

- Examples: $f(x) = \alpha x$; $f(x) = \sqrt{x}$; $f(x) = x^2 + 1$.

Examples 2

- With 2 inputs and 1 output, a typical production function looks like:

$$y \leq f(x_1, x_2),$$

which we can represent in the 2-dimensional input space (*isoquants*!).

- Examples: $f(x_1, x_2) = \min\{x_1, x_2\}$; $f(x_1, x_2) = x_1 + x_2$; $f(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$.

Property 1.

Property 1. Impossibility of free production.

$f(0,0) \leq 0$

Property 2.

Property 2. Possibility of inaction.

$0 \leq f(0,0)$

Input requirement set and q-isoquant.

Define the "**input requirement set (for output y)**" as follows:

$$Z(y) \equiv \{(x_1, x_2) | y \leq f(x_1, x_2)\} \tag{1}$$

Formally, the **y-isoquant**:

$$\{(x_1, x_2) | y = f(x_1, x_2)\} \tag{2}$$

Property 3.

Property 3. Free disposal.

For each $y \in \mathbb{R}_+$, if $x_1' \geq x_1$, $x_2' \geq x_2$, and $y \leq f(x_1, x_2)$, then $y \leq f(x_1', x_2')$.

Properties 4 and 5.

Property 4. Convexity of the input requirement set.

For each $y \in \mathbb{R}_+$, each pair $(x_1, x_2), (x_1', x_2') \in Z(y)$, and each $t \in [0,1]$, it holds that $t(x_1, x_2) + (1-t)(x_1', x_2') \in Z(y)$.

Property 5. Strict convexity of the input requirement set.

For each $y \in \mathbb{R}_+$, each pair $(x_1, x_2), (x_1', x_2') \in Z(y)$, and each $t \in (0,1)$, it holds that $t(x_1, x_2) + (1-t)(x_1', x_2') \in \text{Int}Z(y)$.

Marginal product of input i.

- The **marginal product** of an input $i = 1, 2$ describes the marginal increase of $f(x_1, x_2)$ when marginally increasing x_i.

- Mathematically, this can be written as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1},$$

when $\Delta x_1 \rightarrow 0$. If ϕ is differentiable, the marginal product is the derivative of f w.r.t. x_i evaluated at (x_1, x_2) and is denoted by $MP_i(x_1, x_2)$.

Technical rate of substitution.

- The **technical rate of substitution** (**TRS**) of input i for input j (at **z**) is defined as:

$$TRS\left(x_1, x_2\right) \equiv \frac{\Delta x_2}{\Delta x_1}, \tag{3}$$

such that production is unchanged.

- By first order approximation,

$$\Delta y \cong MP_1 \Delta x_1 + MP_2 \Delta x_2 = 0,$$

  solving, this gives:

$$TRS\left(x_1, x_2\right) = -\frac{MP_1\left(x_1, x_2\right)}{MP_2\left(x_1, x_2\right)}$$

- It reflects the relative value of the inputs (in terms of production) and corresponds to the slope of the y-isoquant at $\left(x_1, x_2\right)$.

Properties 6 and 7.

Property 6. Homotheticity.

For each (x_1, x_2) and each $t > 0$, it holds that $TRS(x_1, x_2) = TRS(tx_1, tx_2)$.

Property 7. Homogeneity of degree r.

For each (x_1, x_2) and each $t > 0$, it holds that $f(tx_1, tx_2) = t^r f(x_1, x_2)$.

Properties 8, 9, and 10.

Property 8. Increasing returns to scale (IRTS).

For each (x_1, x_2) and each $t > 1$, it holds that $f(tx_1, tx_2) > tf(x_1, x_2)$.

Property 9. Decreasing returns to scale (DRTS).

For each (x_1, x_2) and each $t > 1$, it holds that $f(tx_1, tx_2) < tf(x_1, x_2)$.

Property 10. Constant returns to scale (CRTS).

For each (x_1, x_2) and each $t > 0$, it holds that $f(tx_1, tx_2) = tf(x_1, x_2)$.

The optimization problem

- We split the optimization problem of the firm in two parts:

1. Cost minimization (choosing (x_1, x_2) for given y);
2. Output optimization (choosing y, given the cost-minimizing input choices).

The cost minimization problem

- Let quantity $y \in \mathbb{R}_+$ be the output that a firm wants to bring to the market.
- The firm wants to minimize the cost of producing y. How to do it?
- graphically....
- Algebraically. Solve the following minimization problem:

$$\begin{array}{rl} \min_{x_1,x_2} & w_1x_1 + w_2x_2 \\ s.t. & y \leq f(x_1, x_2) \end{array}$$

The Lagrangian and FOCs

$$\mathscr{L}(x_1, x_2, \lambda; w_1, w_2, y) = w_1 x_1 + w_2 x_2 + \lambda (y - f(x_1, x_2)) \tag{4}$$

- The FOCs (allowing for corner solutions!) require that:

$$\lambda^* MP_i (x_1^*, x_2^*) \leq w_i \qquad \text{for } i = 1, 2 \tag{5}$$

$$y \leq f(x_1^*, x_2^*) \tag{6}$$

The Lagrangian and FOCs

- Thus, if $x_i^* > 0$ (implying that $\lambda^* MP_i(x_1^*, x_2^*) = w_i$), a necessary condition for cost minimization is that:

$$\frac{MP_j(x_1^*, x_2^*)}{MP_i(x_1^*, x_2^*)} \leq \frac{w_j}{w_i} \tag{7}$$

- or (for interior solutions): **TRS** equals **input price ratio**.

Conditional demand and cost function

- The **conditional demand function** for input i is:

$$x_i^* = H^i(w_1, w_2, y) \tag{8}$$

- Substituting these conditional demands in the cost minimization problem, we get the relationship between the total cost and the input prices w and the output choice q. This **cost function** is defined by:

$$C(w_1, w_2, y) \equiv w_1 x_1^* + w_2 x_2^* = w_1 H^1(w_1, w_2, y) + w_2 H^2(w_1, w_2, y) \tag{9}$$

Exercise: cost minimization problem (1)

- Determine the cost function for the firm with production function $f(x_1, x_2) = (x_1 x_2)^{\frac{1}{3}}$.
- The minimization problem is:

$$\begin{aligned} \min_{x_1, x_2} \quad & w_1 x_1 + w_2 x_2 \\ s.t. \quad & q \leq \phi(x_1, x_2) = (x_1 x_2)^{\frac{1}{3}} \end{aligned}$$

- Write the Lagrangian:

$$\mathscr{L}(x_1, x_2, \lambda; w_1, w_2, y) = w_1 x_1 + w_2 x_2 + \lambda \left(y - (x_1 x_2)^{\frac{1}{3}} \right)$$

Exercise: cost minimization problem (2)

- The FOCs are:

$$\begin{cases} \lambda^* MP_1(x_1^*, x_2^*) \leq w_1 \\ \lambda^* MP_2(x_1^*, x_2^*) \leq w_2 \\ y \leq (x_1^* x_2^*)^{\frac{1}{3}} \end{cases}$$

- Since f is increasing in x_1 and x_2 and $x_1, x_2 \neq 0$ (WHY?):

$$\begin{cases} \lambda^* \frac{1}{3} (x_1^*)^{-\frac{2}{3}} (x_2^*)^{\frac{1}{3}} = w_1 \\ \lambda^* \frac{1}{3} (x_1^*)^{\frac{1}{3}} (x_2^*)^{-\frac{2}{3}} = w_2 \\ y = (x_1^* x_2^*)^{\frac{1}{3}} \end{cases}$$

Exercise: cost minimization problem (3)

- Dividing the first by the second FOC (and taking the cubic power of the third one), gives:

$$\begin{cases} \frac{x_2^*}{x_1^*} = \frac{w_1}{w_2} \\ y^3 = x_1^* x_2^* \end{cases}$$

- And, solving for x_2^*:

$$x_2^* = \frac{w_1}{w_2} x_1^* = \frac{w_1}{w_2} \frac{y^3}{x_2^*}$$

- Thus:

$$(x_2^*)^2 = y^3 \frac{w_1}{w_2}$$

- and the conditional demand function of input 2 is:

$$x_2^* = H^2(w_1, w_2, y) = y^{\frac{3}{2}} \sqrt{\frac{w_1}{w_2}}$$

Exercise: cost minimization problem (4)

- Since $x_2^* = \frac{w_1}{w_2} x_1^*$, substituting $x_2^* = y^{\frac{3}{2}} \sqrt{\frac{w_1}{w_2}}$ gives the conditional demand function of input 1:

$$x_1^* = H^1(w_1, w_2, y) = y^{\frac{3}{2}} \sqrt{\frac{w_2}{w_1}}$$

- The cost function is defined as:

$$C(w_1, w_2, y) \equiv w_1 x_1^* + w_2 x_2^* = w_1 H^1(w_1, w_2, y) + w_2 H^2(w_1, w_2, y)$$

- Thus, substituting:

$$C(w_1, w_2, y) = w_1 y^{\frac{3}{2}} \sqrt{\frac{w_2}{w_1}} + w_2 y^{\frac{3}{2}} \sqrt{\frac{w_1}{w_2}}$$

- And, simplifying,

$$C(w_1, w_2, y) = 2\sqrt{y^3 w_1 w_2}.$$

Properties of the cost function

- Increasing in all input prices and strictly increasing in at least one; if f is continuous, then also strictly increasing in output y.
- The cost function is homogeneous of degree 1 in prices, i.e. changing all prices by 10% increases total cost by 10%.
- The cost function is concave in input prices.
- [Shephard's Lemma] $\frac{\partial C(w_1,w_2,y)}{\partial w_i} = x_i^* = H^i(w_1, w_2, q)$, i.e. the cost increase when marginally changing the input price is exactly the compensated input demand!

The output optimization problem

- Now that we know how a firm chooses inputs for production, we are left with the following problem:

$$\max_{y \in \mathbb{R}_+} py - C(w_1, w_2, y) \tag{10}$$

- The first order conditions are:

$$\begin{cases} p = C_y(w_1, w_2, y^*) & \text{if } y^* > 0 \\ p < C_y(w_1, w_2, y^*) & \text{if } y^* = 0 \end{cases} \tag{11}$$

- The second order condition is:

$$C_{yy}(w_1, w_2, y^*) \geq 0 \tag{12}$$

Furthermore...

- Our firm needs to be aware that even when profits are maximized, these might not be positive... so we should further require that $\Pi \geq 0$ or:

$$py - C(w_1, w_2, y) \geq 0 \tag{13}$$

or that average cost is lower than p ($\frac{C(w_1, w_2, y)}{y} \leq p$).

Demands and supply functions

- We can define the firm's **supply function** as the relationship between the optimal quantity produced and the market prices of inputs and output:
$$y = S(w_1, w_2, p) \tag{14}$$
- Remember that we already defined the *conditional demand function* for input i as:
$$x_i = H^i(w_1, w_2, y) \tag{15}$$
- We can now substitute (14) in (15) to obtain the **unconditional demand function** for input i:
$$x_i = D^i(w_1, w_2, p) \equiv H^i(w_1, w_2, S(w_1, w_2, p)) \tag{16}$$

Outline

Slope of the supply function

- When $y^* > 0$, the FOC for the output optimization problem requires that:

$$p = C_y(w_1, w_2, y^*)$$

- Substituting the supply function for $y^* = S(w_1, w_2, p)$ gives:

$$p = C_y(w_1, w_2, S(w_1, w_2, p))$$

- Now take the derivative wrt p:

$$1 = C_{yy}(w_1, w_2, S(w_1, w_2, p))\, S_p(w_1, w_2, p)$$

- Rearrange and obtain:

$$S_p(w_1, w_2, p) = \frac{1}{C_{yy}(w_1, w_2, S(w_1, w_2, p))} \geq 0 \qquad (17)$$

- Thus, the slope of the supply function is positive! Why? by the SOC...

Slope of the supply function

- When $y^* > 0$, the FOC for the output optimization problem requires that:

$$p = C_y(w_1, w_2, y^*)$$

- Substituting the supply function for $y^* = S(w_1, w_2, p)$ gives:

$$p = C_y(w_1, w_2, S(w_1, w_2, p))$$

- Now take the derivative wrt p:

$$1 = C_{yy}(w_1, w_2, S(w_1, w_2, p)) S_p(w_1, w_2, p)$$

- Rearrange and obtain:

$$S_p(w_1, w_2, p) = \frac{1}{C_{yy}(w_1, w_2, S(w_1, w_2, p))} \geq 0 \qquad (17)$$

- Thus, **the slope of the supply function is positive**! Why? by the SOC...

Output price effect on input demand

- Consider the uncompensated demand for input $x_i^* = D^i(w_1, w_2, p)$ and take the derivative wrt **output** price p. **Remember** that $D^i(w_1, w_2, p) \equiv H^i(w_1, w_2, S(w_1, w_2, p))$.

$$D_p^i(w_1, w_2, p) = H_y^i(w_1, w_2, y^*)\, S_p(w_1, w_2, p)$$

- By the Shephard's Lemma, $\frac{\partial C(w_1,w_2,y)}{\partial w_i} = H^i(w_1, w_2, y)$. Thus $H_y^i(w_1, w_2, y) = \frac{\partial\left(\frac{\partial C(w_1,w_2,y)}{\partial w_i}\right)}{\partial y} = \frac{\partial C_y(w_1,w_2,y)}{\partial w_i}$ (cross derivatives are equal!). Substituting in the previous gives:

$$D_p^i(w_1, w_2, p) = \frac{\partial C_y(w_1, w_2, y^*)}{\partial w_i} S_p(w_1, w_2, p) \tag{18}$$

- How does uncompensated demand change with output price? If w_i increases the marginal cost of output, then an increase of the output price would imply a larger use of input i.

Input price effect on input demand (1)

- Consider the uncompensated demand for input $x_i^* = D^i(w_1, w_2, p)$ and take the derivative wrt **input** price w_j. (Again, start from the identity $D^i(w_1, w_2, p) \equiv H^i(w_1, w_2, S(w_1, w_2, p))$).

$$D^i_j(w_1, w_2, p) = H^i_j(w_1, w_2, y^*) + H^i_y(w_1, w_2, y^*) S_j(w_1, w_2, p)$$

- As before, by the Shephard's Lemma, $\frac{\partial C(w_1, w_2, y)}{\partial w_i} = H^i(w_1, w_2, y)$. Thus $H^i_y(w_1, w_2, y) = \frac{\partial\left(\frac{\partial C(w_1, w_2, y)}{\partial w_i}\right)}{\partial y} = \frac{\partial C_y(w_1, w_2, y)}{\partial w_i}$ (cross derivatives are equal!).

- Furthermore, differentiate the FOC $p = C_y(w_1, w_2, S(w_1, w_2, p))$ wrt w_j to obtain:

$$0 = \frac{\partial C_y(w_1, w_2, y^*)}{\partial w_j} + C_{yy}(w_1, w_2, y^*) S_j(w_1, w_2, p)$$

Input price effect on input demand (2)

- Substitute to get

$$D_j^i(w_1, w_2, p) = H_j^i(w_1, w_2, y^*) - \frac{C_{iy}(w_1, w_2, y^*)\, C_{jy}(w_1, w_2, y^*)}{C_{yy}(w_1, w_2, y^*)} \quad (19)$$

- How does uncompensated demand change with the price of another input? Two effects: a **substitution effect** $H_j^i(w_1, w_2, y^*)$ and an **output effect** $\frac{C_{iy}(w_1, w_2, y^*) C_{jy}(w_1, w_2, y^*)}{C_{yy}(w_1, w_2, y^*)}$.

Implication 2

- Look now at the effect of w_i on the demand of input i.

$$D_i^i(w_1, w_2, p) = H_i^i(w_1, w_2, q^*) - \frac{[C_{iy}(w_1, w_2, y^*)]^2}{C_{yy}(w_1, w_2, y^*)} \qquad (20)$$

- $H_i^i(w_1, w_2, y) = C_{ii}(w_1, w_2, y)$ (by Shephard's Lemma and taking the derivative).
- By concavity of the cost function (SOC for an optimum), $C_{ii}(w_1, w_2, y^*) \leq 0$. Thus, $H_i^i(w_1, w_2, y^*) \leq 0$.
- But $C_{yy}(w_1, w_2, y^*) \geq 0$ (again from the SOC) and also the squared term is larger than 0; thus:
- $D_i^i(w_1, w_2, p) \leq 0$, i.e. the unconditional demand for input i is decreasing in the own price.

Many products, many inputs...

- Up to now, we have studied the case of a firm producing a single output y. What if the firm could produce many goods at the same time?
- Abstractly, all commodities (inputs or outputs) could be produced. So, let us write a (large) vector $\mathbf{y} \equiv (y_1, ..., y_n) \in \mathbb{R}^n$ of all commodities.
- Then good y_n is a net output if $y_n > 0$; it is net input if $y_n > 0$.

Production technology and MRT

- We can now write the technology as an implicit inequality:

$$F(\mathbf{y}) \leq 0 \tag{21}$$

where the function F is non-decreasing in each of the y_i.

- We define the **marginal rate of transformation** of netput i into netput j by:

$$MRT_{ij} \equiv \frac{MF_j(\mathbf{y})}{MF_i(\mathbf{y})} \tag{22}$$

Objective of the firm

- Our firm still wants to maximize profits (now much simplified):

$$\Pi = \sum_{i=1}^{n} p_i y_i \tag{23}$$

subject to $F(\mathbf{y}) \leq 0$.

- Proceeding as before, we can write the Lagrangean of the maximization problem:

$$\mathscr{L}(\mathbf{y}, \lambda; \mathbf{p}) \equiv \sum_{i=1}^{n} p_i y_i - \lambda F(\mathbf{y}) \tag{24}$$

Optimality conditions

- Deriving wrt each y_i and λ, we get the following FOCs:

$$p_i \geq \lambda^* F_i \left(\mathbf{y}^* \right) \qquad \text{for each } i = 1, \ldots, n \tag{25}$$

$$F\left(\mathbf{y}^* \right) \leq 0 \tag{26}$$

- If $y_i^* > 0$, for each j the following holds at the optimum:

$$\frac{MF_j\left(\mathbf{y}^* \right)}{MF_i\left(\mathbf{y}^* \right)} \leq \frac{p_j}{p_i} \tag{27}$$

- or, equivalently, **MRT equals output price ratio**.

The netput and profit functions

- As before we can write the optimal choice of y_i as a function of the prices: $y_i^* \equiv y_i(\mathbf{p})$.
- Subsituting these netput functions in the profit, we get the profit function:

$$\Pi(\mathbf{p}) \equiv \sum_{i=1}^{n} p_i y_i^* = \sum_{i=1}^{n} p_i y_i(\mathbf{p}) \tag{28}$$

Properties of the profit function

- Non-decreasing in all net-put prices.
- The profit function is homogeneous of degree 1 in prices, i.e. changing all prices by 10% increases total cost by 10%.
- The profit function is convex in net-put prices.
- [Hotelling's Lemma] $\frac{\partial \Pi(\mathbf{p})}{\partial p_i} = y_i^*$, i.e. the marginal profit increase for marginally changing the netput price is exactly the optimal quantity of netput i!