allocation $\mathbf{x}'$. Will this movement increase welfare? If $\mathbf{x}'$ is close to $\mathbf{x}$, we can apply a Taylor series expansion to get

$$W(u_1(\mathbf{x}_1'), \ldots, u_n(\mathbf{x}_n')) - W(u_1(\mathbf{x}_1), \ldots, u_n(\mathbf{x}_n)) \approx \sum_{i=1}^{n} a_i \mathbf{D}u_i(\mathbf{x}_i)(\mathbf{x}_i' - \mathbf{x}_i).$$

Since $(\mathbf{x}, \mathbf{p})$ is a market equilibrium, we can rewrite this as

$$W(u_1(\mathbf{x}_1'), \ldots, u_n(\mathbf{x}_n')) - W(u_1(\mathbf{x}_1), \ldots, u_n(\mathbf{x}_n)) \approx \sum_{i=1}^{n} a_i \lambda_i \mathbf{p}(\mathbf{x}_i' - \mathbf{x}_i).$$

We see that the welfare test reduces to examining a weighted change of expenditures. The weights are related to the value judgments which were originally incorporated into the welfare function.

As a special case, suppose that the original allocation $\mathbf{x}$ is a welfare optimum. Then the results of Chapter 17, page 331, tell us that $\lambda_i = 1/a_i$. In this case we find

$$W(u_1(\mathbf{x}_1'), \ldots, u_n(\mathbf{x}_n')) - W(u_1(\mathbf{x}_1), \ldots, u_n(\mathbf{x}_n)) \approx \sum_{i=1}^{n} \mathbf{p}(\mathbf{x}_i' - \mathbf{x}_i).$$

The distribution terms drop out—since distribution is already optimal—and we are left with a simple criterion: a small project increases welfare if national income (at the original prices) increases. This is exactly the criterion relevant to the compensation test.

This means that if the social planner consistently follows a policy of maximizing welfare both with respect to lump sum income distribution and with respect to other policy choices that affect allocations, then the policy choices that affect the allocations can be valued independently of the effect on the income distribution.

## 22.3 Optimal taxation

We saw in Chapter 8, page 118, that a lump-sum income tax is always preferable to an excise tax. However, in many cases lump-sum taxes are not feasible. What do optimal taxes look like if we are unable to use lump sum taxes?

We examine this question in a one-consumer economy. Let $u(\mathbf{x})$ be the consumer's direct utility function and $v(\mathbf{p}, m)$ be his indirect utility function. We interpret $\mathbf{p}$ as the producer prices. If $\mathbf{t}$ is the vector of taxes, then the price vector faced by the consumer is $\mathbf{p} + \mathbf{t}$. This yields the consumer a utility of $v(\mathbf{p} + \mathbf{t}, m)$ and yields the government a revenue of $R(\mathbf{t}) = \sum_{i=1}^{k} t_i x_i(\mathbf{p} + \mathbf{t}, m)$.

The optimal taxation problem is to maximize the consumer's utility with respect to the tax rates, subject to the constraint that the tax system raises some given amount of revenue, $R$:

$$\max_{t_1,\ldots,t_k} v(\mathbf{p}+\mathbf{t},m)$$

$$\text{such that } \sum_{i=1}^{k} t_i x_i(\mathbf{p}+\mathbf{t},m) = R.$$

The Lagrangian for this problem is

$$\mathcal{L} = v(\mathbf{p}+\mathbf{t},m) - \mu\left[\sum_{i=1}^{k} t_i x_i(\mathbf{p}+\mathbf{t},m) - R\right].$$

Differentiating with respect to $t_i$, we have

$$\frac{\partial v(\mathbf{p}+\mathbf{t},m)}{\partial p_i} - \mu\left[x_i + \sum_{j=1}^{k} t_j \frac{\partial x_j(\mathbf{p}+\mathbf{t},m)}{\partial p_i}\right] = 0 \quad \text{for } i = 1,\ldots,k.$$

Applying Roy's law, we can write

$$-\lambda x_i - \mu\left[x_i + \sum_{j=1}^{k} t_j \frac{\partial x_j(\mathbf{p}+\mathbf{t},m)}{\partial p_i}\right] = 0 \quad \text{for } i = 1,\ldots,k.$$

Solving for $x_i$ we have

$$x_i = -\frac{\mu}{\mu+\lambda}\sum_{j=1}^{k} t_j \frac{\partial x_j(\mathbf{p}+\mathbf{t},m)}{\partial p_i}.$$

Now use the Slutsky equation on the right-hand side of this equation to get

$$x_i = -\frac{\mu}{\mu+\lambda}\sum_{j=1}^{k} t_j \left[\frac{\partial h_j}{\partial p_i} - \frac{\partial x_j}{\partial m} x_i\right].$$

After some manipulation, this expression can be written as

$$\theta x_i = \sum_{j=1}^{k} t_j \frac{\partial h_j}{\partial p_i},$$

where $\theta$ is a function of $\mu$, $\lambda$, and $\sum_j t_j \partial x_j/\partial m$.

Applying the symmetry of the Slutsky matrix, we can write

$$\theta x_i = \sum_{j=1}^{k} t_j \frac{\partial h_i}{\partial p_j}. \tag{22.1}$$

Putting this expression into elasticity form yields

$$\theta = \sum_{j=1}^{k} \frac{\partial h_i}{\partial p_j} \frac{p_j}{x_i} \frac{t_j}{p_j} = \sum_{j=1}^{k} \epsilon_{ij} \frac{t_j}{p_j}.$$

This equation says that the taxes must be chosen so that the weighted sum of the Hicksian cross-price elasticities is the same for all goods.

In the extreme case, where $\epsilon_{ij} = 0$ for $i \neq j$, this condition becomes

$$\frac{t_i}{p_i} = \frac{\theta}{\epsilon_{ii}}. \tag{22.2}$$

so that the tax/price ratio for good $i$ is proportional to the inverse of the elasticity of demand. This is known as the **inverse elasticity rule.** It makes good sense: you should tax goods heavily that are relatively inelastically demanded, and tax goods lightly that are relatively elastically demanded. Doing this distorts the consumer's decisions the least.

Another simplification arises when the tax rates $t_i$ are small. In this case

$$dh_i \approx \sum_{j=1}^{k} t_j \frac{\partial h_i}{\partial p_j}.$$

Inserting this into equation (22.1) gives us

$$\frac{dh_i}{h_i} \approx \theta.$$

This equation says that the optimal set of small taxes reduces all compensated demands by the same proportion.

## Notes

The material in this chapter is pretty standard; consult any text on benefit-cost analysis for elaboration. For a survey of optimal taxation theory see Mirrlees (1982) or Atkinson & Stiglitz (1980).

## Exercises

22.1. In the formula for the optimal tax derived in the text, equation (22.1), show that $\theta$ is nonnegative if the required amount of revenue is positive.

22.2. A public utility produces outputs $x_1, \ldots, x_k$. These goods are consumed by a representative consumer with utility function $u_1(x_1) + \cdots + u_k(x_k) + y$, where $y$ is a numeraire good. The utility produces good $i$ at marginal cost $c_i$ but has fixed costs $F$. Derive a formula for the optimal pricing rule that relates $(p_i - c_i)$ to the elasticity of demand for good $i$.