ECON 3710/4710 Demography of developing countries

STABLE AND STATIONARY POPULATIONS

Lecture note

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Required reading:

- "Stable and stationary models". Chapter 9 in **D. Rowland** *Demographic Methods and Concepts*. Oxford University Press 2003, pp. 300-306, 312-339.
- "Demographic models" Ch. 1 in **United Nations** (1983) *Indirect Techniques for Demographic Estimation* (New York, United Nations). Pages 11-16. **K**
- "Fertility and mortality estimation using model stable age distributions" Ch. 7 in **United Nations** (1983) *Indirect Techniques for Demographic Estimation* (New York, United Nations). Pages 156-166. **K**

Model populations are used to show various links between population variables, such as the numbers of births and deaths, the age distribution, total population size, etc. The intention is to simplify reality, without loosing main characteristics.

Starting point: we consider a closed population, of which the members only experience birth and death. In addition: we break the population down by just two characteristics: age and sex.

Definition: a population is called stable when both its growth rate and its relative age distribution do not change over time.

Hence the size of a stable population grows or diminishes at a constant rate, but each age group has a constant share. The size of each age group grows/diminishes with the same (constant) rate as the size of the whole population.

Property 1: The number of births and the number of deaths in a stable population change with the same rate. This rate is constant, and it equals the growth rate of the population. Thus the crude birth rate (CBR) is also constant, and so is the crude death rate (CDR) – although the CBR and the CDR are usually not equal to each other.

In other words growth rate (= natural growth rate) = CBR – CDR = constant.

Property 2: Any two sets of time-constant age-specific birth rates and age-specific death rates, when applied to a population with a certain initial age distribution, will in the long run result in a stable population, of which the growth rate and the age distribution only depend of the birth rates and the death rates. The ultimate (stable) age distribution will have a regular form (cf. below). This is the case even if the initial age distribution was not stable – for instance was very irregular. We say that a stable population has "forgotten" its history.
This property is known as the principle of strong ergodicity (Lotka (1911), Euler (1760).

The use of stable population theory

Stable population theory is frequently used when we lack certain population data (for example the CBR) while we know some of the other data (for example the age distribution) with certainty. As an example (to be discussed later)
Brazil 1960:

- from the Census of 1960 we know the age distribution in five-year age groups
- we assume that the population in Brazil in 1960 is stable

Now we can compute ("estimate") the CBR, the CDR, the life expectancy, the Gross Reproduction Rate, etc., by using the theoretical mathematical relationships that exist in a stable population between these variables and the age distribution.

Estimation techniques of this kind ("Indirect estimation methods") are frequently used for developing countries, but also for historical populations in developed countries. Cf. computer exercise on Norwegian population census of 1801.

Using the strong ergodicity principle

Given a set of age-specific birth rates and one of age-specific death rates, it is possible to compute the annual growth rate and the relative age distribution of the stable population that these rates imply. The annual growth rate summarizes, in one number, the age-specific birth and death rates.

Definition: the annual growth rate of a stable population is called the intrinsic growth rate.

Note: the intrinsic growth rate is generally NOT the same as the growth rate for an actual population. These two rates are only equal when the actual population is stable (and closed for migration). Otherwise the intrinsic growth rate is merely a theoretical notion.

Given two sets of age-specific rates for fertility and mortality. How do we compute

- the intrinsic growth rate?
- the stable age distribution?

We limit ourselves to the female part of the population. Hence age-specific birth rates are restricted to female births.
Write the intrinsic growth rate as r.
Two approaches will be presented: an approximate one, and an exact one.

1. Approximate calculation of the intrinsic growth rate and the stable age distribution
We use the Net Reproduction Rate (*NRR*), which can be computed on the basis of the age specific rates for fertility and mortality. See Rowland p. 246 (not required reading). On page 305 he writes the *NRR* as R_0. *NRR* represents the average number of daughters per woman.

Below I will derive an approximate expression for the intrinsic growth, which is somewhat easier to understand and simpler to compute than the one that Rowland gives on page 306 (with an example in Table 9.1). Usually the two formulas give results that are very close.

Assume that the average generation distance between mothers and daughters equals T years. Then we know that *NRR* represents the relative increase in the stable population over a period of T years. The annual growth rate in the stable population equals r. Therefore we can write

(1) $$NRR = (1+r)^T$$

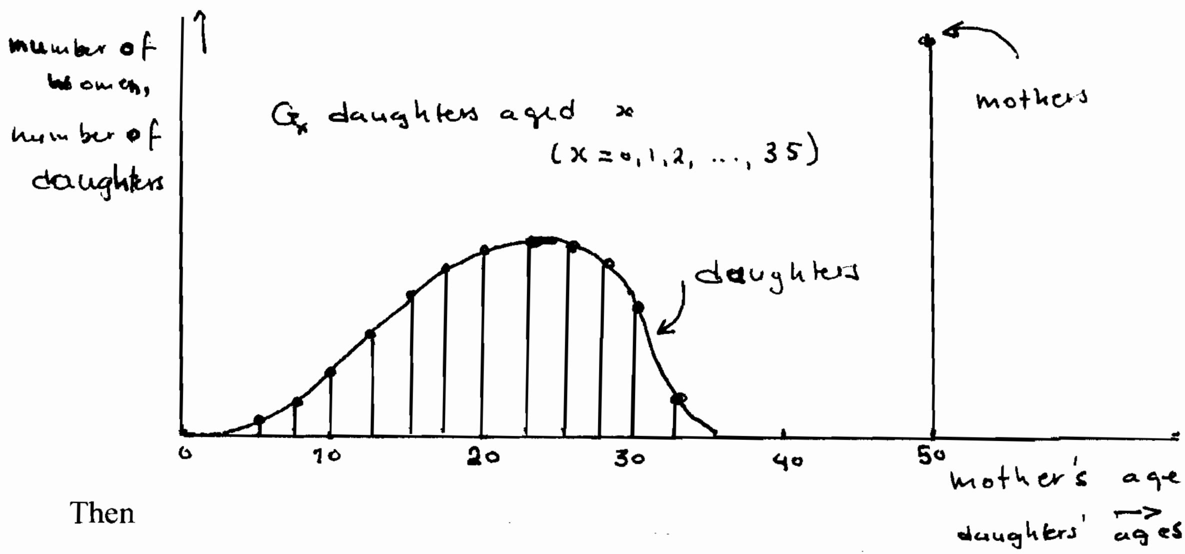
Now we can find r, by solving expression (1):

(2) $$r = \sqrt[T]{NRR} - 1 = (NRR)^{1/T} - 1$$

NRR follows from the age-specific rates. But T?

Consider a group of women at age 50. They have finished their reproductive career.

They have given birth to various numbers of daughters at various ages. When the mothers are 50 years of age, the daughters are between 0 and 35 years old. Define G_x as the number of daughters currently of age x of mothers aged 50, $x = 0, 1, 2, \ldots, 35$.



Then

$$T = \frac{G_0(50-0) + G_1(50-1) + \ldots + G_{35}(50-35)}{G_0 + G_1 + \ldots + G_{35}} =$$

$$= \frac{50(G_0 + G_1 + \ldots + G_{35}) - (0.G_0 + 1.G_1 + \ldots + 35.G_{35})}{G_0 + G_1 + \ldots + G_{35}} =$$

$$= 50 - \frac{0.G_0 + 1.G_1 + \ldots + 35.G_{35}}{G_0 + G_1 + \ldots + G_{35}}.$$

Thus the average generation distance T equals the difference between two ages: the age of the mothers (50) and the *average* age of the daughters. Assume now that this age difference equals the mean age of mothers when they gave birth to these daughters *computed on the basis of the net fertility rates* (R_1/R_0 in Rowland's Table 9.1). The latter mean age is written as μ. See page 6 for the notion of net fertility rate.

We assume that the average generation distance between mothers and daughters is equal to the mean age of the mothers when they give birth to the daughters, in other words we assume that $T = \mu$.

This assumption is only 100 per cent correct provided that $r = 0$, in other words when the stable population neither increases nor diminishes (stationary population, cf. below). In that case the number of women does not change over time, and the mean age computed on the basis of absolute numbers

(T) equals the mean age computed from age-specific rates (μ). For intrinsic growth rates different from zero, is T different from μ ($r > 0$: $\mu > T$; $r < 0$: $\mu < T$).

Therefore, an approximate solution for r is

(3) $$r = \sqrt[\mu]{NRR} - 1 .$$

For example (Bangladesh 1974) when μ=28, and NRR=2.3737, we find that r =1.031 – 1, or 3.1 per cent per year.

After the intrinsic growth rate r has been calculated, we can find the stable age distribution for the female population in two stages. First, compute an initial share of age group (x, x+1) as

$$C_x = (1 + r)^{-(x+½)}.(L_x/l_0).$$

Here L_x and l_0 follow from the life table that is computed from the age-specific death rates: L_x is the column of exposure times, and l_0 is the radix.

The sum of these quantities C_x is not equal to one, as should be the case for a proper share. Assume that this sum equals $C = \sum_x C_x$. Next, compute the final shares $c_x = C_x/C$. For a proof, see the Appendix (not required reading). See also Rowland pages 307-311 (not required reading).
This formula for the age distribution is exact, given r.

2. Exact calculation of the intrinsic growth rate and the stable age distribution
In the Appendix I prove the following exact expression for the intrinsic growth rate r:

(4) $$\sum_x \{(1 + r)^{-(x+½)} . f_x (L_x / l_0)\} = 1 .$$

Here r is expressed as a function of age-specific birth rates f_x (only female births) and age-specific mortality L_x/l_0 (to be computed from a life table, which itself is based on the age-specific death rates m_x for these women). This expression is called Lotka's fundamental equation of stable population theory. You do not have to remember this expression by heart, but you should be able to interpret and use it.

How can we find the intrinsic growth rate r, when we are given age-specific fertility (f_x) and mortality (L_x/l_0)? Expression (4) is too complicated for a direct solution. We find r by "trial and error", as follows. Choose a starting value r_1 for the intrinsic growth rate. Check whether the left-hand side of (4) equals 1 with this value (and given the age-specific fertility and mortality). If not, choose a new value: r_2. Check (4) again … etc., until you have found a value for the intrinsic growth rate that results in a value for the left-hand side of (4) that is close enough to 1.

Systematic search:

- Choose $r_1 = \sqrt[\mu]{NRR} - 1$, in other words, the approximate solution
- Check the sum in the left-hand side of (4) with $r=r_1$: is sum(r_1) = 1?
- If not, choose a new value r_2 for r: $r_2 = r_1 + [\text{sum}(r_1) - 1]/\mu$
- Check now the new sum against one: is sum(r_2) = 1? If not, repeat the previous step as often as is necessary
- $r_3 = r_2 + [\text{sum}(r_2) - 1]/\mu$
- $r_4 = r_3 + [\text{sum}(r_3) - 1]/\mu$
- …
- …
- etc.

Usually, this procedure results in only a few steps in a value for the intrinsic growth rate that is accurate enough, in other words that produces a value for the left-hand side of expression (4) which is close enough to one.
Example: Peru 1961 (Coale). PC-exercise Egypt 1997, Norway 1987.

Comments

1. Fertility rates f_x are zero at all ages below 15 or above 50. Thus in practice, the sum in expression (4) is limited to ages x = 15, 16, 17, …, 49.
2. The product ($f_x \cdot L_x/l_0$) is called the net fertility rate, often written as φ_x. We recognize the sum $\sum_x \varphi_x$ as the Net Reproduction Rate. Indeed, when we assume no growth in the stable population and insert $r = 0$ in expression (4), we find $\sum_x \varphi_x = 1$. Or, the Net Reproduction Rate equals one in this case!
3. When the age x is given in five-year intervals instead of one-year intervals, expression (4) is written as follows:

$$\sum_x \left\{(1+r)^{-(x+2\frac{1}{2})} \cdot {}_5\varphi_x\right\} = 1, \quad {}_5\varphi_x = {}_5f_x({}_5L_x / l_0).$$

where x represents intervals 15-19, 20-24, …, 45-49.

4. As soon as we have computed the exact value of the intrinsic growth rate r, we can also compute the exact value of the generation distance T, as follows:
   we know that $NRR = (1 + r)^T$
   next we can solve for T : $T = \log(NRR)/\log(1 + r)$.
5. The fundamental equation (4) is often written as follows: $\int e^{-rx}.f(x).p(x)dx$. In that case, age is a continuous variable – not a discrete variable (as in our case – one-year age groups, or five-year age groups). Therefore
   - integral sign $\int$, instead of summation $\sum$
   - e^{-rx}, instead of $(1+r)^{-(x+½)}$
   - $p(x)$, instead of L_x/l_0; $p(x)$ equals l_x/l_0, the individual survival probability from birth to exact age x

   At the same time we have that $NRR = e^{rT}$; instead of $NRR = (1 + r)^T$

   and $r = (1/T).\ln(NRR)$ instead of $r = \sqrt[T]{NRR} - 1$.

   Whether you use discrete or continuous age in a stable population model has in general very little effect on the numerical results, unless you use too wide age intervals. Five-year intervals give results that are accurate enough for our purpose.
6. After you have computed the exact value for the intrinsic growth rate, you can compute the stable age distribution using the method described in Section 1.

Read Rowland chapter 9, pages 12-16 in UN (1983), and the example on the use of stable population theory (the case of Brazil 1960) in UN (1983).

Stationary populations

A stationary population is a stable population in which the intrinsic growth rate is zero. In other words, none of the population variables in a stationary population change over time: the annual number of births, the annual number of deaths, population size, the size of a certain age group, etc. are all constant.

We can derive a number of simple expressions for these variables in a stationary population. Here I will mention them, and try to give an intuitive justification. The appendix contains a formal derivation of those expressions. The expressions for stationary populations in this section are required reading, those in the appendix are not.

Fertility

Given age-specific rates for fertility and mortality for women. These rates are combined into net fertility rates $\varphi_x = f_x(L_x/l_0)$. Since a stationary population is a stable population with zero growth rate, we can insert $r = 0$ in expression (4) and find

$$\sum_x \left\{(1+r)^{-(x+½)}.\varphi_x\right\} = \sum_x \varphi_x = 1.$$

We know that the sum $\sum_x \varphi_x$ equals the Net Reproduction Rate. Thus we see that a stationary population has *NRR* equal to 1. We find the same result when we start from the expression $NRR=(1+r)^T$, which holds for every stable population. Inserting r=0 results in $NRR=(1+0)^T=1$. Also the other way round: a closed stable population that has *NRR* equal to one is stationary.

Mortality

The life expectancy at birth e_0 equals the mean age at death of the stationary population. In other words, individuals die at age e_0 on average. The annual number of deaths in a stationary population is constant. Write this number as D. Population size, written as N, is also constant. Each year, a share $1/e_0$ in the population dies. Thus we find that $D = N/e_0$.

This leads to the following expression for the Crude Death Rate (CDR) in a stationary population:

$$\text{CDR} = D/MYP = D/N = 1/e_0,$$

because the mid-year population MYP equals $½(N+N)=N$.

The link between fertility and mortality in a stationary population

In a stable population we have that the growth rate equals the difference between crude birth rate and crude death rate. But in a stationary population the growth rate is zero. Thus we find for a stationary population that

(5) CDR = CBR = $1/e_0$.

Write the annual number of births as B. Since CBR = $B/N = 1/e_0$, we find the following simple expression for total population size in a stationary population:

(6) $N = B.e_0$.

The age structure of a stationary population

The life expectancy e_0 is defined in general as $e_0 = T_0/l_0 = \sum_x L_x/l_0$, see life table theory. Inserting this expression for e_0 in expression (6), we find that total population size in a stationary population equals $B.\sum_x L_x/l_0$. Meanwhile one can prove that the size of age group $(x,x+1)$ in a stationary population equals $B.L_x/l_0$.

B and l_0 do not depend of age x - only L_x varies with age. This is the reason why L_x (the L_x-column in a life table) is sometimes called "age structure of the stationary population". See Rowland page 307.

Appendix: Some mathematics on stable and stationary populations

We analyse a stable population that is closed for migration, and restrict the analysis to women.

The aim is to find expressions for the intrinsic growth rate r and the stable age structure c_x, given sets of age-specific rates for mortality and fertility.

The number of births increases each year by a constant factor $(1+r)$. Write the number of births in year t as B_t. By way of example, we note that $B_{2000} = B_{1999}(1+r) = B_{1998}(1+r)^2 = B_{1997}(1+r)^3 = \ldots$ etc. Generally:

- there are B_t live births in year t - these were born between exact time t and exact time t+1;
- at time t there are N_x persons aged between x and x+1;

Then we can write

$$N_0 = B_{t-1} \cdot \frac{L_0}{l_0} = B_t (1+r)^{-1} \cdot \frac{L_0}{l_0}$$

$$N_1 = B_{t-2} \cdot \frac{L_1}{l_0} = B_t (1+r)^{-2} \cdot \frac{L_1}{l_0}$$

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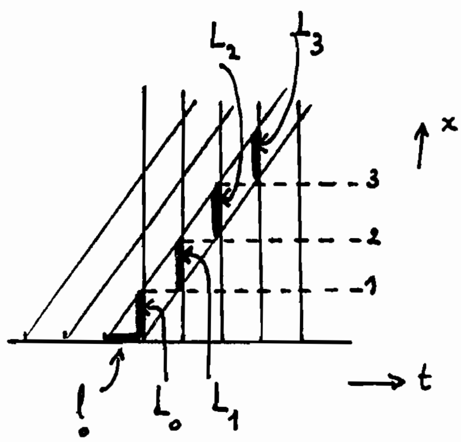
$$N_x = B_{t-x-1} \cdot \frac{L_x}{l_0} = B_t (1+r)^{-(x+1)} \cdot \frac{L_x}{l_0}$$



Thus total population size at time t equals

$$\sum_{x=0}^{\omega} N_x = B_t \sum_{0}^{\omega} (1+r)^{-(x+1)} \frac{L_x}{l_0}.$$

One year later, at time (t+1), population size equals

$$(1+r) B_t \sum_{0}^{\omega} (1+r)^{-(x+1)} \frac{L_x}{l_0}.$$

This implies that the mid-year population in year t can be approximated by

$$MYP_t = (1+r)^{½} B_t \sum_{0}^{\omega} (1+r)^{-(x+1)} \frac{L_x}{l_0},$$

and the Crude Birth Rate in the stable population in year t equals

$$\text{(A1)} \qquad CBR_t = \frac{B_t}{MYP_t} = \frac{1}{\sum_{x=0}^{\omega} (1+r)^{-(x+½)} \cdot \frac{L_x}{l_0}} = b,$$

which is independent of time, see Property 1 in the main text.

Expression (A1) can also be written as follows

$$\text{(A2)}\qquad \frac{1}{\sum_{x=0}^{\omega}(1+r)^{-(x+1)}.\frac{L_x}{l_0}} = b(1+r)^{½}.$$

At the same time we can write for the share of age group (x,x+1) in the total population

$$c_x = \frac{N_x}{\sum_{x=0}^{\omega} N_x} = \frac{B_t(1+r)^{-(x+1)}.\frac{L_x}{l_0}}{\sum_{x=0}^{\omega} B_t(1+r)^{-(x+1)}.\frac{L_x}{l_0}}.$$

The terms B_t in the numerator and denominator cancel, and the remainder of the denominator can be written as $1/\{b(1+r)^{½}\}$, because of expression (A2). Hence we find for the share of age x

$$\text{(A3)}\qquad c_x = b(1+r)^{-(x+½)}.\frac{L_x}{l_0}.$$

The Crude Birth Rate b can be written as a weighted average of the age-specific birth rates f_x, with the age shares c_x as weights: $b = \sum_x f_x . c_x$. Using (A3) this implies that

$$b = \sum_{x=0}^{\omega} f_x b(1+r)^{-(x+½)}.\frac{L_x}{l_0},$$

or

$$\text{(A4)}\qquad 1 = \sum_{x=0}^{\omega}.(1+r)^{-(x+½)}.f_x\frac{L_x}{l_0}.$$

Expression (A4) is known as Lotka's fundamental equation of stable population theory, or the Lotka equation for short. Given age-specific fertility (f_x) and age-specific mortality (L_x/l_0), the intrinsic growth rate r is determined by Lotka's equation.

When we combine expressions (A1) and (A3) we find for the age structure

$$c_x = \frac{(1+r)^{-(x+\frac{1}{2})}.\frac{L_x}{l_0}}{\sum_{x=0}^{\omega}(1+r)^{-(x+\frac{1}{2})}.\frac{L_x}{l_0}},$$

so that, indeed $\sum_x c_x = 1$.

A stationary population is a special case of a stable population - the growth rate r is zero.

1. The Lotka equation reduces to $\sum_x f_x.(L_x/l_0) = \sum_x \varphi_x = NRR = 1$.
2. Crude Birth Rate (b)

   Insert $r = 0$ in (A1) to find $b = \dfrac{1}{\sum_{x=0}^{\omega}(L_x/l_0)} = \dfrac{1}{e_0}$,

   with e_0 the life expectancy at birth.
3. Crude Death Rate (m)

   In a stationary population, the Crude Birth Rate equals the Crude Death Rate. Therefore

   $m = \dfrac{1}{e_0}$.
4. Age distribution (c_x)

   Use (A3) and insert $r = 0$, to find

   $c_x = b.\dfrac{L_x}{l_0} = \dfrac{L_x}{e_0 l_0}$.
5. Population size (N)

   In general, the CBR equals B_t/MYP_t. In a stationary population, the number of births is constant (B_t=B), and so is population size. Hence MYP_t equals N for a stationary population. Thus the Crude Birth Rate is B/N. At the same time, it is equal to $1/e_0$. In other words, $N = B.e_0$.
6. Size of age group (x,x+1)

   $N_x = N.c_x = Be_0 \dfrac{L_x}{e_0 l_0} = B \dfrac{L_x}{l_0}$.