I. RELATIONS INVOLVING THE GROWTH AND AGE STRUCTURE OF POPULATIONS

1

Measures of Growth and Reproduction

Ansley J. Coale

The numerical increase of a population during any period equals the number of births minus the number of deaths plus the number of net immigrants. Dividing these numbers by the number of person-years lived during the period gives the rate of increase, the birth rate, the death rate, and the rate of net migration. The difference between the birth and death rates is called the rate of natural increase. The birth rate, the death rate, and the rate of natural increase are determined by the age composition of a population as well as by the fertility and mortality rates to which it is subject. Specifically, the birth rate is the sum (or the integral) of the product of the proportion of the population at each age consisting of females at that age and the specific rate of childbearing. Similarly the death rate is the sum of the product of proportions by age and age specific mortality rates. Thus the birth rate in one of two populations with the same schedule of fertility rates can be much higher than in the other when the first population has a higher proportion of women in the ages of high fertility. Similarly, the death rate is strongly affected by the proportion of persons at the ages of highest mortality, particularly at the older ages. In many years the death rate of Mexico has been lower than the U.S. rate, although the mortality rate at every age was higher than in the U.S., because the Mexican population had much higher proportions at ages of low mortality. Because of this strong influence on rates of age composition (as opposed to the risks of death and childbearing), population birth rates, death rates and rates of natural increase are referred to as "crude" rates. Various measures of fertility and mortality have been introduced that are independent of age structure. The commonest of these are the total fertility rate and the expectation of life at birth and its reciprocal, the stationary death rate. These measures are derived entirely from age schedules of fertility and mortality, and are independent of the age composition of the population experiencing the rates. The total fertility rate (TFR) is the sum of the single-year fertility rates in a specified period, and is equal to the average total number of children borne by a hypothetical group of women subject through their lives to these rates. The expectation of life at birth is the mean age at death in a hypothetical group, subject between birth and the highest age attained, to the death rates at each age of the period in question. Its reciprocal is the death rate in a "stationary" population, whose age distribution is wholly determined by the mortality schedule.

There are also conventional measures of the rate of reproduction of a female (or a male) population. The logic of these measures is that they indicate the ratio of the number of daughters to the number of mothers (or of one generation to the next earlier) implied by the fertility (or the fertility and mortality) of a given period. The Gross Reproduction Rate (GRR) is a measure of the numbers of daughters per woman that would be borne according to specified rates of bearing daughters to a group of women not subject to mortality. ("Gross" indicates that their is no allowance for the effect of mortality.) The Net Reproduction Rate (NRR) is a measure of the number of daughters per woman that would be borne according to specified rates of mortality and of bearing daughters to a group of women subject through life to these rates. The GRR is a measure of the hypothetical ratio of successive generations with specified fertility and no mortality; the NRR is the ratio with specified fertility *and* mortality. Since there is little evidence of consequential differences according to maternal age in the proportion of births that are female, the GRR equals the TFR multiplied by the fraction (female births)/(total births).

The NRR is useful in developing the concept of the stable population, discussed in detail in the later parts of this chapter. The stable population, a population with a fixed age distribution and a constant birth rate, death rate, and rate of increase, is the population that emerges from the long continuation of any specified combination of a fertility schedule and a mortality schedule. The characteristics of the stable population are fully determined from these two schedules. The NRR is a characteristic of the stable population; as the ratio of daughters to mothers according to prevailing fertility and mortality, it is the ratio of one generation of women to the next preceding: it is the multiplier of the size of the stable population in one generation. The stable population with an annual rate of increase of r is multiplied by e^{rN} in N years, hence it is multiplied by e^{rT} in T years, where T is the mean length of the female generation. The calculation of GRR, NRR, the stable rate of increase, and the mean length of generation are illustrated in the following example, using data for Peru in 1961 (taken from Shryock and Siegel, 1973). (Table 1).

Table 1
Determination of GRR, NRR, Intrinsic Rate of Increase, and Mean Length of Generation

of Age Woman	(1) Rate of Bearing Female Children	(2) ${}_5L_x / l_0$	(3) (1)·(2) Net Fertility	$e^{-ar}\cdot(3)$ r=.01993	r=.020138
15-19	0.0389	3.8870	0.1512	0.1067	0.1063
20-24	0.1134	3.7980	0.4307	0.2751	0.2738
25-29	0.1200	3.6880	0.4425	0.2558	0.2543
30-34	0.0986	3.5730	0.3522	0.1843	0.1831
35-39	0.0748	3.4540	0.2585	0.1224	0.1215
40-44	0.0338	3.3260	0.1123	0.0481	0.0447
45-49	0.0109	3.1800	0.0348	0.0135	0.0134
Sum	(x5) 2.4520 (GRR)	---	1.782 (NRR)	1.0059	1.0000

The rate of increase of the stable population is the value of r that satisfies the equation $\int_0^\infty e^{-ra}p(a)m(a)da=1$, where p(a) is the proportion surviving from birth to age a according to the given mortality schedule, and m(a) is the proportion of women at age a, giving birth to a female child. In this example r is determined by a simple process of successive approximation. An approximate value of r is substituted in the integral given above, which is evaluated numerically. If the value of r were correct, the integral would equal 1.0. The error in the value of the integral provides an adjustment to the approximation of r, because if y is the value of the integral, differentiation shows that dy/dr equals -A, where A is the mean age of childbearing in the stable population. Hence a slightly erroneous r yields an integral that differs from 1.0 by -A times the error in r, and the correct value of r can be estimated as the first approximation plus the difference of the integral from 1.0, divided by A. The correct value of r is also (ln(NRR))/T; but the algebraic determination of T is complicated. The process of successive approximation applied to the integral equation for r can proceed by assigning the crude figure of 29 years to both A and T (which are not in fact exactly equal to each other). Thus the first approximation of r is (lnNRR)/29, or .01993; when this value is used in the integral expression, the sum is 1.0059 instead of 1.0.. The second approximation is .01993 plus the error in the value of the integral (.0059) divided by 29 (as a crude approximation of the mean age of maternity in the stable population), or .020133. With this r the integral is 1.00015, and the calculation of r is surely more precise than the population data on which it as based. (Further iteration yields an r of .020138 giving an integral equal to 1.0 to six decimal places, a clearly redundant refinement). When the value of r has been thus determined, the birth rate, death rate and age composition of the stable age distribution can be calculated from equations in a later section of this chapter, entitled "stable population theory."

Editor's Note:
Relations in Populations with Fixed and Variable Rates of Fertility, Mortality, and Migration

An early, complete and integrated treatment of stable populations, *The Concept of the Stable Population*, was published by the United Nations in 1966 (Demographic Studies No. 39). Its principal author, Jean Bourgeois-Pichat, was at the time a member of the staff of the Population Division. Because it is a long document, it could not be included in these readings. Because it uses its own internally consistent but unique terminology, it is not practical to include excerpts. The following reading has been used in its place.

2

Stable Population Theory

Ansley J. Coale

"Stable Population." *The New Palgrave: A Dictionary of Economics*, vol. 4. London: MacMillan Press, 1987, pp. 466-69.

stable population theory. Many years ago A.J. Lotka (1911) proved that a population (of one sex; for simplicity this discussion will be restricted to females) not gaining or losing by migration, and subject to an unchanging age-schedule of death rates and rates of childbearing, has an age distribution, birth rate, death rate, and rate of increase that do not change. All of these fixed characteristics are determined by the mortality and fertility schedules to which the population is subject. Lotka called such a population stable, using the term in a technical sense borrowed from physics; if the population is perturbed by a momentary change in fertility or mortality, 'stability' implies that it returns after a while to its equilibrium state of constant birth rate, death rate, and age structure.

Lotka's proof of stability implies that a closed population experiencing fixed mortality and fertility schedules arrives at a fixed and determinate age structure, no matter what arbitrary and irregular age distribution and population had at an early point. This property of converging to a fixed form was labelled 'strong ergodicity' by John Hajnal (1958). Strong ergodicity means that when fixed rates have long prevailed, the unchanging age structure of the stable population is independent of its form at any much earlier time; figuratively, it can be said that a stable population forgets its past.

More than forty years later, Coale (1957) made the conjecture that all human populations forget their past. Obviously, when fertility and mortality schedules constantly change, the age structure of the population constantly changes. The changing age structure is nevertheless independent of the remote past. The age distribution of France is no longer much affected by excess mortality and reduced numbers of births during the Napoleonic wars, and the age distribution of Greece is no longer affected at all by the Peloponnesian Wars. The independence of a changing age distribution from long past influences is called 'weak ergodicity'. Any population, whether or not stable, has forgotten the remote past; the stable population, in addition to forgetting the past, has a fixed form, and fixed birth and death rates. A mathematical proof of the weak ergodicity of human population was provided by Alvaro Lopez (1961).

BASIC EQUATIONS OF THE STABLE POPULATION. A proof of weak ergodicity when population density is treated as a continuous function of age and time (Lopez, 1967) provides a convenient background for the equations that characterize a stable population.

In any closed population, the number of persons at age a at time t is

$$N(a,t) = B(t-a)p(a,t), \tag{1}$$

when $B(t)$ is the number of births at time t, and $p(a,t)$ is the proportion surviving from birth to age a of those born at time $t-a$. The number of births, in turn is determined as follows:

$$B(t) = \int_\alpha^\beta N(a,t)m(a,t)\,\mathrm{d}a, \tag{2}$$

where $m(a,t)$ is the proportion of women at age a at time t bearing a female child, and α and β are the lower and upper limits of the age-span in which childbearing occurs.

Lopez proved that age structure is independent of the remote past by showing that the birth sequences in two populations subject to the same succession of mortality and fertility schedules approach a constant ratio one to the other, no matter how different the two populations may be at some initial moment. In other terms, the ratio $B_1(t)/B_2(t)$ approaches a constant K as populations 1 and 2 remain subject to the same changing sequences of fertility and mortality schedules.

Let $\gamma(t) = B_1(t)/B_2(t)$. It follows from equations (1) and (2) that

$$B_1(t) = \int_\alpha^\beta B_1(t-a)p(a,t)m(a,t)\,\mathrm{d}a, \tag{3}$$

and that $B_2(t)$ conforms to the same equation with a change only in subscript. But $B_1(t-a) = \gamma(t-a)B_2(t-a)$; hence

$$B_1(t) = \int_\alpha^\beta \gamma(t-a)B_2(t-a)p(a,t)m(a,t)\,\mathrm{d}a. \tag{4}$$

Since $B_1(t)/B_2(t)$ is $\gamma(t)$,

$$\gamma(t) = \int_\alpha^\beta \gamma(t-a)\{B_2(t-a)p(a,t)m(a,t)/B_2(t)\}\,\mathrm{d}a. \tag{5}$$

The expression in brackets in equation (5) is the proportionate distribution by age of mother, $B_2(t)$, of the births at time t in the second population. Thus the expression in brackets is a frequency distribution, $f(a,t)$, summing to 1.0 when added over ages α to β. Hence equation (5) can be rewritten as

$$\gamma(t) = \int_\alpha^\beta \gamma(t-a)f(a,t)\,\mathrm{d}a. \tag{6}$$

The ratio of $B_1(t)$ to $B_2(t)$ is the weighted average of the sequence of ratios of $B_1(t)$ to $B_2(t)$ α and β years in the past. Continued application of such averaging over many generations ultimately brings the ratio $B_1(t)/B_2(t)$ to a constant. When the ratio has been constant for ω years (ω the highest age attained) and mortality in the two populations is the same, the ratio $N_1(a,t)/N_2(a,t)$ is also the same at all ages; the populations differ in size, but have the same proportionate age composition. If two populations with arbitrarily different initial conditions come to have the same age distribution when subject to the same sequence of fertility and mortality schedules, they may be said to have forgotten the past. The full proof (not repeated here) of weak ergodicity includes a formal demonstration of the intuitively appealing proposition that repeated averaging by a continuous weighting function with positive values over a finite range leads to a constant value of the ratio. The essential feature, then, of human fertility that leads to weak ergodicity is the prevalence in all large populations of positive fertility rates in an extended span of ages. If, on the contrary, fertility were concentrated at a single age, there would be no averaging, no convergence of $B_1(t)/B_2(t)$ to a constant, and no 'forgetting' of the remote past.

Strong ergodicity is an immediate corollary of weak ergodicity. If a population experiences unchanging fertility and mortality schedules for a long time, this year's history is the same as last year's. Two populations with the same history of fertility and mortality have the same age distribution. It follows that unchanging fertility and mortality produce an unchanging age distribution – the age distribution of a stable population.

Let $c(a)\mathrm{d}a$ be the proportion of the stable population in the age interval a to $a+\mathrm{d}a$; then $c(a) = N(a)/\int_0^\omega N(a)\,\mathrm{d}a$, where ω is the highest age attained. In any female population, the birth-rate is $b = \int_0^\omega c(a)m(a)\,\mathrm{d}a$, and the death-rate is $d = \int_0^\omega c(a)\mu(a)\,\mathrm{d}a$, where $\mu(a)$ is the death-rate at age a. Since the age distribution in a population with fixed fertility and mortality schedules is unchanging, it follows that the birth-rate and death-rate do not change. Hence the rate of increase r (which equals $b-d$) is fixed.

An exact expression for the unchanging age distribution of a stable population is implied by the mortality and fertility schedules to which it is subject. The formula for the age distribution of a stable age distribution is derived as follows. The proportion at age a, $c(a)$, is defined as $N(a,t)/N_T(t)$, where $N_T(t)$ is the total population at time t. But $N(a,t) = B(t-a)p(a,t)$, and $B(t-a) = b\cdot N_T(t-a)$. Moreover, $N_T(t-a) = N_T(t)e^{-ra}$; hence $c(a) = bN_T(t)e^{-ra}p(a)/N_T(t)$, or

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Since $\int_0^\omega c(a)\,\mathrm{d}a = 1.0$, it follows that

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There remains the determination of r. In any female population the birth rate is determined by the age distribution and the age schedule of bearing female children; that is, $b = \int_\alpha^\beta c(a)m(a)\,\mathrm{d}a$. Thus $b = b\int_\alpha^\beta e^{-ra}p(a)m(a)\,\mathrm{d}a$, from which it follows that

$$\int_\alpha^\beta e^{-ra}p(a)m(a)\,\mathrm{d}a = 1.0. \tag{9}$$

Equation (9) provides the means for calculating the rate of increase r in the stable population by successive approximation, given the maternity schedule, $m(a)$, and the mortality schedule, $\mu(a)$. [The proportion surviving, $p(a)$ equals $\int_0^a e^{-\mu(x)}\,\mathrm{d}x$.] The numerical value of the integral in equation (9) is a monotonically decreasing function of r; the integral for any specified value of r can be determined by standard numerical methods; and trial and error can quickly find the value of r that causes the integral to equal 1.0. When r is known, b can be calculated from equation (8), $c(a)$ from equation (7).

The description to this point of stable population theory is expressed in terms of fertility and mortality schedules and age distributions that are continuous functions of age. The theory has alternatively been formulated with distributions and schedules expressed as discrete variables. A population distributed in discrete age intervals at a given moment can be considered a vector that is transformed into the ensuing population vector through multiplication by a transition matrix, the Leslie matrix (Leslie, 1945). The terms 'weak' and 'strong' ergodicity were first applied to finite Markov chains (Hajnal, 1958); weak ergodicity as a property of populations was first proved by employing matrix algebra and discrete age distributions (Lopez, 1961).

USE OF STABLE POPULATED CONCEPTS AND RELATIONS BEFORE LOTKA. Mathematicians, actuaries and demographers made use of the characteristics of a stable population long before Lotka's discovery of what Hajnal called strong ergodicity.

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A century and a half earlier Leonard Euler (1760) worked out many of the relations that characterize stable populations. He postulated a population subject to two hypotheses; the hypothesis of mortality, by which Euler meant a fixed life table [or $p(a)$ in the terminology employed here), and the hypothesis of multiplication (or constant value of r]. He noted that the hypothesis of mortality is an assumption that the regime of mortality remains ever the same, and that the hypothesis of multiplication is equivalent to an assumption of a constant birth-rate. Euler's equations treat age and time as discrete variables, with one-year intervals, rather than as continuous variables. Translated into continuous notation, one of his equations is $N = B\int_0^\infty e^{-ra}p(a)$, where N is the total population, and B the number of births in a given year. This equation is equivalent to equation (8) above. Euler notes that if the life table is known, the growth rate can be calculated from the birth-rate, or the birth-rate from the growth rate. He also shows that $N(a) = B\,e^{-ra}p(a)$ [equivalent to equation (7)], and derives a number of other equations, including an expression for the distribution of deaths by age in a population described by his two hypotheses.

Euler's work seems to have been little noted in subsequent years except by actuaries, one of whom, Joshua Milne (1815), cited Euler and developed himself a full set of equations for a constantly growing population with a fixed life table. Like Euler, Milne used integral values of age and time, and expressed growth over x years at the rate r as $(1+r)^x$, but explained in a footnote that a 'logarithmic expression' would be more precise. Milne is of special interest to demographers because he made calculations to help Malthus in Malthus's preparation of an essay entitled *Population*, published in 1824 in the Supplement to the Fourth Edition of the *Encyclopaedia Britannica*. In this essay Malthus (with Milne's help) constructed a stable population from a life table borrowed from Sweden and Finland and a rate of increase that causes the populations to double every 25 years. He showed that the age distribution of this stable population closely matched the distribution recorded in the United States in 1800, 1910 and 1820; and supported his hypothesis that by natural increase alone the American population was growing at such a rapid rate (Coale, 1979).

ANALYTICAL USES OF STABLE POPULATIONS. In a single-volume summary of his contributions to the mathematics of population, Lotka (1939) devoted many pages to the analysis of constantly growing populations with a fixed schedule of mortality. Only in later chapters did he introduce the relations (including stability) that incorporate a schedule of fertility rates by age of mother. Apparently unaware of the earlier work by Euler and Milne, he designated populations that are subject to an unchanging life table and grow at a constant rate *Malthusian populations*; not, evidently, because he knew of the use of the mathematics of such populations in Malthus's last essay on population, but rather because of Malthus's well-known belief that populations tend to grow at a geometric rate unless checked.

The incorporation of a schedule of rates of childbearing in addition to a mortality schedule into the mathematical analysis of population has been quite useful, not merely because it permits the proof of stability (strong ergodicity). A fundamental and analytically useful feature of stable population theory is that the combination of any schedule of fertility with any schedule of mortality connotes a population with a specific age structure, birth rate, death rate, and rate of natural increase. The implied population may never exist but it nevertheless is implied in full calculable detail.

For example, equations (7) to (9) can be used to determine the characteristics of the population that would be generated by a combination of the highest observed (or highest imaginable) fertility in a human population with the lowest observed (or lowest imaginable) mortality rates. Some highly fertile populations have recorded rates of childbearing that would yield 8.0 to 8.5 children ever born by women who reach age 50 subject to these rates. Other populations have recently attained female expectations of life approaching 80 years. The stable population generated by a combination of this high fertility and this low mortality would have a rate of increase of 49.5 per thousand. If all women survived to age 100, the rate of increase would be very slightly higher (49.7 per thousand), the birth rate would be 50.0 per thousand, and the death rate 0.3 per thousand. Higher fertility than in the above example has been observed among married women in some populations. If marriage were universal by age 15, if the widowed remarried immediately, and if married women experienced these very high marital fertility rates, the mean number of children born by age 50 would be about 12. With an expectation of life at birth of 80 years, this still higher fertility would yield a stable population with a birth rate 65.9 per thousand, a death rate of 1.1 per thousand, and a rate of increase of 64.8 per thousand. Again, a mortality schedule with no deaths below age 100 would generate a stable population with a slightly higher rate of increase (65.0 per thousand).

Another application of the stable population inherent in a combination of a fertility schedule and a mortality schedule is the comparison of the characteristics of such a stable population with the characteristics of the actual population that experiences the fertility and mortality in question. The age distribution of the actual population is determined by its past experience. Its age distribution in combination with its current fertility and mortality schedules, determines its birth and death rates. A comparison of actual characteristics with stable population characteristics shows how different the population would be if shaped by current birth and death experience rather than by its past.

Table 1 illustrates this use of stable analysis.

In 1941 the observed birth-rate was higher, and the observed death-rate lower, than actual rates, because history had created an age distribution with a higher proportion in the reproductive ages, and a lower proportion in the old ages, where mortality is higher, than in the stable. Note the negative rate of increase, high proportion over 65, and high mean age ultimately implied by the low fertility of 1941. In 1963 the contrasts between actual and stable population are the opposite.

SOME PRACTICAL USES OF STABLE POPULATION MATHEMATICS. The constant fertility and constant mortality that would establish a stable population are certainly not universal features of the

TABLE 1. Vital Rates and Age Distribution of Actual and Stable Female Populations Compared, England and Wales, 1941 and 1963

	Rates per thousand persons						Percent in age interval					
	Birth		Death		Natural increase		0–14		65+		Mean age	
Year	O	S	O	S	O	S	O	S	O	S	O	S
1941	13.1	10.3	11.8	20.5	1.3	−10.2	19.6	15.5	10.3	19.9	35.9	42.4
1963	17.2	20.0	11.6	9.2	5.6	10.8	21.4	27.1	14.1	11.0	37.7	33.2

'O' means Observed population; 'S' the Stable.
Source: Keyfitz, N. and Flieger, W. (1968).

history of actual populations. By the middle of the 20th century, the mortality of most populations had fallen; in the more industrialized countries, at least, fertility was much reduced from earlier levels. Nevertheless, the equations relating fertility, mortality, growth, and age composition in stable populations have proven highly useful in estimating the true characteristics of some populations for which accurate and complete data are lacking.

The usefulness of stable population theory in making good estimates from faulty data originates in a phenomenon called 'quasi-stability'. A quasi-stable population is one in which fertility has in recent years been approximately constant, but mortality has steadily declined for one or two decades, or more. Such a population has an age distribution little different from the stable distribution implicit in current fertility and mortality schedules. It could be said, metaphorically, that trends in mortality are almost completely forgotten as they occur (Bourgeois-Pichat, 1958; Coale, 1962). Because of the resemblance of the age distribution of a quasi-stable population to the currently implied stable age distribution, the equations of stable population mathematics can be used to obtain approximate values of the birth-rate and other measures of fertility, and of the death-rate and other measures of mortality, from a recorded age distribution and an estimated rate of increase. A history of constant fertility and recently declining mortality was characteristic of many less developed countries in the 1950s and 1960s. Such application of stable population analysis occupies all of a report written by Bourgeois-Pichat for the United Nations (1968b), and is a major theme in two other manuals on estimation published by the United Nations (*Manual IV*, 1968, and *Manual X*, 1983).

In recent years it has been shown (Bennett and Horiuchi, 1981; Preston and Coale, 1982; Arthur and Vaupel, 1984) that the question relating the age distribution of a stable population to its constant rate of increase and to the fixed mortality schedule to which it is subject can be modified slightly to apply to any population, in particular to a closed population in which fertility and mortality have recently varied rather than remaining unchanged. The equation for the age distribution of a stable population

$$c(a) = b\,e^{-ra}p(a)$$

is modified to

$$c(a) = b \exp\left[-\int_0^a r(x)\,\mathrm{d}x\right]p(a),$$

where $p(a)$ is now an expression of the proportion that would survive to age a according to the mortality schedule at the moment (or during the period) for which $c(a)$ is the proportion at age a. The exponential factor now incorporates the sum of the growth rates, $[r(x)]$, that vary with age (rather than a times a fixed rate r) over the range from 0 to a. This extension provides a much more flexible basis for estimation, and doubtless will replace stable population analysis in most such uses.

Progenitors of stable population theory existed at least 150 years before the concept of stability was invented, and its validity proven. The theory promises to have descendants for many years in the future, non-stable descendants that will doubtless have as much abstract and practical value as the stable theory itself.

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