On an example from lecture 2007-11-14

First a note on concave functions: of course I managed to give (graphically) the definition of *convex* functions in the lecture. A function is concave if for any two points a and b on the graph, the *straight line-segment from a to b*, is nowhere above the graph. So yes, linear functions are concave – they are the only functions that are both concave and convex.

To the problem: Consider the (linear programming) maximization

$$\max -x-y \qquad \text{subject to} \begin{cases} p_2 - p_1 - 2x - y \leq 0 & \text{(C1)} \\ p_3 - p_1 - x - 2y \leq 0 & \text{(C2)} \\ x \leq p_2 & \text{(C3)} \\ y \leq p_3 & \text{(C4)} \\ -x \leq 0 & \text{(C5)} \\ -y \leq 0 & \text{(C6)} \end{cases}$$

where

$$p_3 > p_2 > p_1 \geq 0. \tag{*}$$

The constraints define a set S in the (x, y) plane. You are given the following problem:

> *If conditions (C1) & (C2) hold with equality for some point $(x_0, y_0) \in S$ with $x_0 > 0$, $y_0 > 0$, then show that (x_0, y_0) satisfies the sufficient conditions (i.e. the necessary Kuhn-Tucker conditions together with concavity of the Lagrangian).*

The Lagrangian is then

$$\begin{aligned} L(x,y) = -x - y - \lambda_1(p_2 - p_1 - 2x - y) - \lambda_2(p_3 - p_1 - x - 2y) \\ - \lambda_3(x - p_2) - \lambda_4(y - p_3) + \lambda_5 x + \lambda_6 y \end{aligned}$$

and the Kuhn-Tucker conditions are

$$\begin{aligned} 0 &= L'_x && && \text{(KT1)} \\ 0 &= L'_y && && \text{(KT2)} \\ \lambda_1 &\geq 0 && (= 0 \text{ if strict inequality in (C1)}) && \text{(KT3)} \\ \lambda_2 &\geq 0 && (= 0 \text{ if strict inequality in (C2)}) && \text{(KT4)} \\ \lambda_3 &\geq 0 && (= 0 \text{ if strict inequality in (C3)}) && \text{(KT5)} \\ \lambda_4 &\geq 0 && (= 0 \text{ if strict inequality in (C4)}) && \text{(KT6)} \\ \lambda_5 &\geq 0 && (= 0 \text{ if strict inequality in (C5)}) && \text{(KT7)} \\ \lambda_6 &\geq 0 && (= 0 \text{ if strict inequality in (C6)}) && \text{(KT8)} \end{aligned}$$

Long, isn't it? Well, we can get rid of two already: we are told that constraints (C5) and (C6) are inactive, so $\lambda_5 = \lambda_6 = 0$. Also we are told that (C1) and (C2) hold with equality, so (x_0, y_0) solves

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} p_2 - p_1 \\ p_3 - p_1 \end{pmatrix}$$

which by Cramer's rule gives

$$x_0 = (2p_2 - 2p_1) - (p_3 - p_1))/3 = (-p_3 + 2p_2 - p_1)/3 \quad \text{and}$$
$$y_0 = (2p_3 - 2p_1 - (p_2 - p_1))/3 = (2p_3 - p_2 - p_1)/3$$

We are given the information that $(x_0, y_0) \in S$, so that (C3) and (C4) hold. Now we can do one out of two:

- We can check whether conditions (C3) and (C4) are inactive.
- We can guess (from the sketch) that $\lambda_3 = \lambda_4 = 0$ – it *may* be necessary if (C3) resp. (C4) hold with strict inequality – and then insert this into the Kuhn-Tucker conditions and hope that they hold. (That's the good thing about sufficient conditions: you can guess first and check afterwards.)

Let us check. Those who feel lucky, can skip this paragraph: To check (C3), we observe that if (C3) fails – i.e. that $x_0 > p_2$ – then we must have $-p_3 + 2p_2 - p_1 > 3p_2$ or $0 > p_1 + p_2 + p_3$, which is impossible by (*). So (C3) holds. To check (C4), we observe that if (C4) fails – i.e. that $y_0 > p_3$ – then we must have $2p_3 - p_2 - p_1 > 3p_3$, with the same conclusion.

OK, so both the checkers and the guessers among you would now want to put $\lambda_3 = \lambda_4 = 0$. Since (C1) and (C2) both hold with equality, then what is left of the Kuhn-Tucker conditions is that there must exist nonnegative numbers λ_1 and λ_2 such that (x_0, y_0) is a stationary point of L.

To verify this, put $0 = L'_x = -1 + 2\lambda_1 + \lambda_2$ (since $\lambda_3 = \cdots = \lambda_6 = 0$) and $0 = L'_y = -1 + \lambda_1 + 2\lambda_2$. So

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and Cramer's rule gives $\lambda_1 = (2-1)/3 \geq 0$ and $\lambda_2 = (2-1)/3 \geq 0$. We see that the point (x_0, y_0) satisfies the Kuhn-Tucker conditions with $\lambda_1 = \lambda_2 = 1/3$, $\lambda_3 = \cdots = \lambda_6 = 0$.

Since the criterion and the constraints are all linear functions – and hence concave – the sufficient conditions hold.