University of Oslo / Department of Economics/ Nils Framstad / 2007–10–02

Note: On the solution to the first compulsory term paper in ECON3120/4120 Mathematics 2, fall 2007

This final version is updated with a few words on the grading and a few comments on each problem, based on what turned out to be the main difficulties.

On the grading (see the reverse side of the front matter for your grades and scores):

- The grading scale has *not* been adjusted to reflect the difficulties of the problem set. (Problem 2d is deleted completely though, as it contained an error.) However, one paper with a score that would normally fail on an exam, was nevertheless accepted. The grade distribution for the other 28 accepted papers turned out as $(A, B, C, D, E) = (1, 9, 16, 1, 1)$.
- I have given each main problem 1,. . . ,5 equal weight, and within each main problem, the letter-enumerated subproblems are equally weighted.
- The fact that everything is scored in 12ths (because 12 is the least common denominator of 3 and 4, so I can use both quarters and thirds – very scientific, eh?), does not mean that the accuracy is $1/12$; although the common mistakes are hopefully punished consistently, I have been less accurate than for a proper exam.
- Scores for Problem 5 are however very rough guesstimates. That problem turned out harder than I thought.

Problem 1 We consider the function $f(x) = 2x^2 - \ln x - 2, \quad x > 0$. Both for points (a) and (b), we could use the derivative $f'(x) = 4x - 1/x = (4x^2 - 1)/x$; since $x > 0$, the sign is the same as the sign of $4x^2 - 1$, which is first negative then zero in one point ($x = 1/2$), then positive.[1]

(a) We want to show that $f(x)$ has exactly one other zero, and that this point is in the interval $(0, 1)$.

- As mentioned, we have f' first negative, then zero at one point, then positive. So f is strictly decreasing to a minimum (which turns out to be $x = 1/2$), and strictly increasing from there.
- On the strictly decreasing part («small x») there is at most one zero, and on the strictly increasing part («large x») there is at most one zero (draw a diagram!). So f has no more than two zeros, and we know that there is at least one, namely $x = 1$.
- So the proof will be complete if we can show that (i) f starts out positive near $x = 0$, and (ii) is negative for some $x_0 < 1$; then we know from the intermediate value theorem («skjæringssetningen») that there is one zero z in $(0, x_0)$, and we will also know that the only zeroes are z and 1.
  (i) $f(0^+) = +\infty$. OK.

[1] The other positive part of this quadratic polynomial occurs on the negative half-axis, where f is not defined.

(ii) There are various ways to prove that there is an $x_0 < 1$ for which $f < 0$. One obvious way is to find the point where $f' = 0$, namely $x_0 = 1/2$, and insert and calculate. Maybe a bit more elegant is to show that $f'(1) > 0$, which is done by direct calculation.

(b) From the previous discussion, we have that that $f'(x)$ has the same sign as $4x^2 - 1$, which is > 0 on $(1, \infty)$ (it is even > 3). So f is strictly increasing, and hence one-to-one, on $(1, \infty)$, so it has an inverse g there. To find $g(0)$, observe that $f(1) = 0$, so that $g'(0) = 1/f'(1) = 1/3$; if you don't remember this formula, you know that g is defined through the formula $g(f(x)) = x$, and differentiating using the chain rule yields $g'(f(x)) \cdot f'(x) = 1$, hence $g'(f(x)) = 1/f'(x)$. Now insert $x = 1$ to get $\underline{\underline{g'(1) = 1/3}}$.

Comments problem 1: In point (a), there are three things to prove: existence of some zero other than one, uniqueness (that there is no more than one such!), and where it is. The latter was easy. A lot of you did not provide sufficient proof of existence – the intermediate value theorem («skjæringssetningen») is key here. And a few of you used that, but did not point out that there is *at most* one zero in each interval.
In point (b), most of you could point out that strict monotonicity is sufficient to prove existence of inverse. However, quite a lot of you failed to put the right x and y into the formula $g'(y) = 1/f'(x)$. If that is hard, write $x = g(f(x))$ as above.

Problem 2

(a) To differentiate the equation system

$$\begin{aligned} x \cdot y \cdot z \cdot ue^{-u} \cdot ve^{v} &= 0 \\ 1x + 2y + 3z + 4u + 5v &= 6 \end{aligned}$$

we first observe that $\frac{d}{dw}(we^{\pm w}) = e^{\pm w} \pm we^{\pm w} = e^{\pm w}(1 \pm w)$, useful for the differentiation wrt. u and v. We proceed to get the following answer:

$$\underline{\underline{\begin{cases} yzue^{-u}ve^{v}\,dx + xzue^{-u}ve^{v}\,dy + xyue^{-u}ve^{v}\,dz & \\ +xyze^{-u}(1-u)ve^{v}\,du + xyzue^{-u}e^{v}(1+v)\,dv & = 0 \\ dx + 2dy + 3dz + 4du + 5dv & = 0 \end{cases}}}$$

(b) First, we note that we can simplify the first equation by dividing by the common nonzero factor $e^{-u}e^{v}$. To find a general expression for v'_x, we also note that the dy and dz terms will not be interesting; rewrite the differentiated equation system into:

$$\begin{aligned} xyz(1-u)v\,du + xyzu(1+v)\,dv &= -yzuv\,dx + P\,dy + Q\,dz0 \\ 4du + 5dv &= -dx - 2dy - 3dz \end{aligned}$$

We want to solve this for dv, that is, eliminate du from the equations. Multiply the first equation by 4 and the second by $xyz(1-u)v$, to get:

$$\begin{aligned} 4xyz(1-u)v\,du + 4xyzu(1+v)\,dv &= -4yzuv\,dx + 4P\,dy + 4Q\,dz \\ 4xyz(1-u)v\,du + 5xyz(1-u)v\,dv &= -xyz(1-u)v\,dx - 2xyz(1-u)v\,dy - xyz(1-u)v\,dz \end{aligned}$$

Subtract and write (e.g.) R and S for the uninteresting dy and dz coefficients:

$$(5xyz(1-u)v - 4xyzu(1+v))\, dv = (4yzuv - xyz(1-u)v)\, dx + R\, dy + S\, dz$$

or

$$dv = \frac{(4yzuv - xyz(1-u)v)\, dx + R\, dy + S\, dz}{(5xyz(1-u)v - 4xyzu(1+v))}$$

v'_x is now the dx coefficient:

$$\begin{aligned} v'_x &= \frac{4yzuv - xyz(1-u)v}{5xyz(1-u)v - 4xyzu(1+v)} \\ &= \frac{v}{x}\frac{4u - x + xu}{5v - 5uv - 4u - 4uv)} \\ &= \underline{\underline{\frac{v}{x}\frac{4u - x + xu}{5v - 9uv - 4u}}} \end{aligned}$$

(c) We want to show that if P has (x, y, z)-coordinates equal to $(1, 1, 1)$, show that $u(1, 1, 1) = v(1, 1, 1) = 0$. With $(x, y, z) = (1, 1, 1)$, then the first equation of the set is $uv = 0$ – so that either $u = 0$ or $v = 0$ – while the second is $4u + 5v = 0$.

- If $u = 0$, then $5v = 0$ so that $v = 0$.
- If $v = 0$, then $4u = 0$ so that $u = 0$.
- So in both cases, we have $u = v = 0$, QED.

(d) This is the embarrassing point on my behalf, and you may disregard the following if you are not interested. Anyway, to explain what goes wrong: with the coefficients chosen, the problem is not possible to solve by our tools; with $(x, y, z, u, v) = (1, 1, 1, 0, 0)$, the boxed differentiated equation set above becomes

$$\begin{aligned} 0\, dx + 0\, dy + 0\, dz + 0\, du + 0\, dv &= 0 \\ 4\, du + 5, dv &= -(dx + 2dy + 3dz) \end{aligned}$$

which does not have any unique solution in for du, since the first equation $0 = 0$ gives no information whatsoever.

Indeed, the problem is impossible to solve – not by our tools, not by *any* tools. The original system becomes

$$\begin{aligned} uv &= 0 && \text{(I)} \\ 4u + 5v &= 6 - (x + 2y + 3z) \\ &= -0.3 \quad \text{with } (x, y, z) = (1, 1, 1) && \text{(II)} \end{aligned}$$

This equation set has not a unique solution; to get from $(x, y, z) = (1, 1, 1)$ to $(x, y, z) = (1, 1, 1.1)$, we can either keep u fixed and $= 0$ and move v (continuously!) to $v = 0.3/5 = 0.06$ – or we can keep v constant $= 0$ and move u (continuously) to $u = 0.3/4 = 0.0075$. So the postulated setting of the problem set fails: the equation set does define u, v as continuously differentiable functions of (x, y, z) around some «suitable» point, but $(1, 1, 1, 0, 0)$ is *not* such a suitable point.

Comments problem 2: Differentiating (Norwegian: dere skal «differensiere» systemet, ikke «derivere») involved some fairly lengthy calculations, but turned out easy – I hope, I might have overlooked some errors. But you did very well on that point.
(b): this is solvable by cookbook (see note on course website), but some of you need more practice. Some of you could also benefit from being a bit more aware what the question is – in this case it was v_x, which means you should get rid of du, and save yourselves the work of solving for it.
(c): This wasn't too hard.
(d): Only a few of you attempted to do this. I am not sure whether that was because you tried and failed, or you read the all-too-late note on the website before you attempted, or the problem is that you would not have been able to solve a similar properly-specified problem.

Problem 3

(a) To calculate $\lim_{x\to 0} \dfrac{\sqrt{x+3}-\sqrt{3}}{\sqrt{x+2}-\sqrt{2}}$, we may first observe that it is a «$\frac{0}{0}$» expression, and use l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{\sqrt{x+2}-\sqrt{2}} = «\frac{0}{0}» = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{x+3}}}{\frac{1}{2\sqrt{x+2}}} = \frac{\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{2}}} = \underline{\underline{\frac{1}{3}\sqrt{6}}}$$

(b) The key to calculating $\lim_{x\to 0^+} x^{1/1000}(\ln x)^{9999}$, is the exposition of the following limit given in the the lecture, which is also given in EMEA p. 265, «An Important Limit» (MA I p. 224, «En viktig grenseverdi»):

$$\text{For fixed numbers } p,\ a, \text{ with } a>1,\ \lim_{x\to\infty} \frac{x^p}{a^x} = 0$$

– that is, exponential decay eventually kills polynomial growth.

To solve the problem at hand, you may choose any of the following approaches:

- Put $y = \ln x$ so that $x = e^y$. Then the limit becomes

$$\lim_{y\to -\infty} y^{9999} \cdot e^{y\cdot 1/1000} = \lim_{y\to -\infty} \frac{y^{9999}}{(e^{1/1000})^{-y}}$$

  – note that y tends to *negative* infinity. Now apply the result from the book, to get $\underline{\underline{0}}$.
- Alternatively, note that $\lim_{x\to 0^+} x^{1/1000}(\ln x)^{9999} = \big(\lim_{x\to 0^+} x^{1/9999000}(\ln x)\big)^{9999}$ (because the function of raising something to a power (like 9999), is continuous – and continuous functions are precisely the ones we can interchange with limits). This limit

is

$$\Big(\lim_{x\to 0^+}\frac{\ln x}{x^{-1/9999000}}\Big)^{9999} = \big(«\frac{\infty}{\infty}»\big)^{9999}$$
$$= \Big(\lim_{x\to 0^+}\frac{1/x}{\frac{-1}{9999000}x^{-1/9999000-1}}\Big)^{9999}$$
$$= \big(-9999000\lim_{x\to 0^+} x^{1/9999000}\big)^{9999}$$
$$= \underline{\underline{0}}.$$

This is the method used in EMEA – but not in MA I, which uses a different proof.

- Finally, the method I used – maybe a bit more informal, but without the trick: $\lim_{x\to 0^+}\frac{(\ln x)^q}{x^{-1/1000}}$ will be an «∞/∞» expression as long as $q>0$. Applying l'Hôpital once, we get
$$\lim_{x\to 0^+}\frac{q(\ln x)^{q-1}\frac{1}{x}}{\frac{-1}{1000}x^{-1/1000}\frac{1}{x}} = -1000q\lim_{x\to 0^+}\frac{(\ln x)^{q-1}}{x^{-1/1000}}.$$
What we have achieved, is to reduce the exponent from q to $q-1$, but we have gotten a constant in front. The expression is still «∞/∞» as long as the exponent is strictly positive, so when $q=9999$, we can apply l'Hôpital's rule a total of 9999 times to get
$$K\cdot\lim_{x\to 0^+}\frac{(\ln x)^0}{x^{-1/1000}} = \underline{\underline{0}}.$$

Any of the methods work.

(b) To find $\int xe^{\sqrt{x}}\,dx$, substitute $u=\sqrt{x}\ \Rightarrow\ du=\frac{dx}{2\sqrt{x}}\ \Rightarrow 2u\,du = dx$ to get

$$\int u^2e^u 2u\,du = \int 2u^3e^u\,du$$

which we integrate by parts three times, using $f=$ the monomial and $g'=$ the exponential each time:

$$= 2u^3e^u - \int 6u^2e^u\,du$$
$$= 2u^3e^u - (6u^2e^u - \int 12ue^u\,du)$$
$$= 2u^3e^u - 6u^2e^u + 12ue^u - \int 12e^u\,du$$
$$= e^u\big(2u^3 - 6u^2 + 12u - 12\big) + C$$
$$= \underline{\underline{e^{\sqrt{x}}\big(2\sqrt{x}^3 - 6x + 12\sqrt{x} - 12\big) + C}}$$

(c) We have $f(x) = \displaystyle\int_0^\infty t^x e^{-t}\,dt$, defined for $x\geq 0$. Note that there is an improper integral, defined to be the limit of $\int_0^R$ as $R\to\infty$ (when the limit exists!)

To calculate $f(0)$, we need to calculate $\int_0^\infty e^{-t}\,dt = \lim_{R\to\infty}\big[-e^{-t}\big]_{t=0}^{t=R} = -\lim_{R\to\infty} e^{-R} + 1 = \underline{\underline{1}}$.

To prove that $f(x+1) = (x+1)f(x)$, we integrate by parts, using $u = t^{x+1}$ and $v' = e^{-t}$:

$$\begin{aligned} f(x+1) &= \int_0^\infty t^{x+1}e^{-t}\,dt \\ &= \lim_{R\to\infty}\int_0^R t^{x+1}e^{-t}\,dt \\ &= \lim_{R\to\infty}\Big(\big[-t^{x+1}e^{-t}\big]_{t=0}^{t=R} - \int_0^R (x+1)\cdot t^{x+1-1}\cdot(-e^{-t})\,dt\Big). \end{aligned}$$

The limit of the latter integral is $(x+1)f(x)$, so the proof will be complete if we can prove $\lim_{R\to\infty}\big[-t^{x+1}e^{-t}\big]_{t=0}^{t=R} = 0$. Since $0^{x+1}e^{-0} = 0$, we only need to show $\lim_{R\to\infty} R^{x+1}e^{-R} = 0$. Again, this is the important limit in EMEA p. 265 (MA I page 224), and we are done.

Comments problem 3: Point (a) turned out easy, as expected.
Point (b) was expected to be awkward, and it was. But it is very close to something which was pointed out as important both in the book and on a lecture – and which you also needed for the final point in Problem 3. A few of you tried some quite hairy shortcuts, like claiming that this has to be zero because one of the terms tends to zero.
The second point (b) (yes they were erroneously enumerated) turned out good for most of you – as expected, as the substitution is fairly obvious. But *please remember that $+C$ term!*
As for the last point, this served as to check whether you could perform a dt integration with an x in there without being confused (most of you passed that test), to check if you are well aware what an improper integral is (well many of you did write $\infty e^{-\infty}$ without noticing that it is supposed to be a limit) – and it checked whether you could find a valid argument for $t^{x+1}e^{-t} \to 0$ as $t \to \infty$. Very few of you did. The erroneous short cuts from point (b) above were far more numerous here, indicating that many of you did not recognize this as a limit at all. For an exam, I should most probably have skipped the first point (b), and re-written the last point into something like «Show that $t^{x+1}e^{-t} \to 0$ as $t \to \infty$, and use this to ...»

Problem 4 The differential equation $t\dot{x} = x^3 \ln t$ is separable. $x(t) = 0$ for all t is clearly a solution; for $x \neq 0$ we can write the differential equation as

$$\begin{aligned} x^{-3}dx &= \frac{\ln t}{t}dt \\ \Rightarrow \qquad \int x^{-3}dx &= \int \frac{\ln t}{t}dt \end{aligned}$$

Substituting $u = \ln t$, we get $du = dt/t$, and

$$-\tfrac{1}{2}x^{-2} = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}(\ln t)^2 + C$$

so that the general solution is

$$x(t) = \begin{cases} (K - (\ln t)^2)^{-1/2}, & \text{any } K \in (-\infty, \infty), \text{ or} \\ -(K - (\ln t)^2)^{-1/2}, & \text{any } K \in (-\infty, \infty), \text{ or} \\ 0 \end{cases}$$

where $K = -C$ (yes the C might be negative! How many of you found it troublesome to take the square root of $-C - (\ln t)^2$?)

Comments problem 4: *None* of you found the constant solution. I suggest that you check for constant solutions first: Whenever you encounter a separable differential equation, before you separate (i.e. divide by the x^3), ask yourselves «when am I not allowed to divide by x^3?», and you will pick up points which are really too good to miss.
Apart from that, you fared well on this problem. But: have I mentioned the $+C$? Yes I have. And please put it in when you are done *integrating*, don't wait until you have solved for x ...
... which is something you also should do! In real life, it might not always be possible, but in this course you should do it, even if it gives you another chance to make a mistake (like I did, when I forgot the negative solution. So I haven't punished you for not seeing it.)

Problem 5 For points (b) and (c): You can use that $\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3}$ even for $x < 0$ (that is: the left and right hand sides are both well-defined, and they represent the same value).

(a) The scope of integration (the interval $[-12, 12]$) is symmetric around the origin, and so is the integrand $x^{\text{odd number}} \cdot e^{x^{\text{even number}}}$ itself (in other words, it is an «odd function»). So the area under the curve from 0 to 12 will exactly match the area over the curve from -12 to 12. Hence the answer is zero.

For those of you who want a proper proof, a function f is symmetric about the origin if $f(-x) = -f(x)$. Then split the integral:

$$\int_{-k}^{k} f(x)dx = \int_{0}^{k} f(x)dx + \int_{-k}^{0} f(x)\,dx.$$

Now substitute $y = -x$ in the latter integral; then $f(x) = f(-y) = -f(y)$ (the latter by symmetry), and $dy = -dx$, while the $-k$ limit becomes k and the upper limit 0 remains 0. For a *proper* integral, the following holds:

$$\begin{aligned} \int_{0}^{k} f(x)dx + \int_{-k}^{0} f(x)\,dx &= \int_{0}^{k} f(x)dx + \int_{k}^{0} (-f(y))\,(-dy) \\ &= \int_{0}^{k} f(x)dx + \int_{k}^{0} f(y)\,dy \qquad (*) \\ &= F(k) - F(0) + F(0) - F(k) = \underline{0} \end{aligned}$$

where $F' = f$.

(b) For improper integrals, we have to check convergence. There are different kinds of improper integrals (infinite interval/unbounded function); these are two where the integrand is unbounded near an interior point c in the set we integrate over, and we *have to check convergence near* c. In the calculations given, that is not done; the argument is therefore flawed no matter whether the conclusion is correct or not.

(c) We are given that an antiderivative in (i) is $\frac{3}{2}x^{2/3}$, and we also know that an antiderivative to $1/x$ is $\ln|x|$.

(i) For this integral, have

$$\begin{aligned}
\int_{-1}^{1} x^{-1/3}\,dx &= \lim_{R\to 0^-}\int_{-1}^{R} x^{-1/3}\,dx + \lim_{S\to 0^+}\int_{S}^{1} x^{-1/3}\,dx \\
&= \lim_{R\to 0^-}\left[\tfrac{3}{2}x^{2/3}\right]_{-1}^{R} + \lim_{S\to 0^+}\left[\tfrac{3}{2}x^{2/3}\right]_{S}^{1} \\
&= \frac{3}{2}\cdot\big(\lim_{R\to 0^-} R^{2/3} - (-1)^{2/3} + 1^{2/3} - \lim_{S\to 0^+} S^{2/3}\big) \\
&= \underline{\underline{0}}.
\end{aligned}$$

So the answer is, indeed, 0, but the argument given was insufficient.

(ii) We have

$$\begin{aligned}
\int_{-2}^{2} x^{-1}\,dx &= \lim_{R\to 0^-}\int_{-2}^{R} x^{-1}\,dx + \lim_{S\to 0^+}\int_{S}^{2} x^{-1}\,dx \\
&= \lim_{R\to 0^-}\Big[\ln|x|\Big]_{-2}^{R} + \lim_{S\to 0^+}\Big[\ln|x|\Big]_{S}^{2} \\
&= \lim_{R\to 0^-}\ln|R| - \ln 2 + \ln 2 - \lim_{S\to 0^+}\ln S
\end{aligned}$$

which <u>does not exist</u> since $\ln 0^+ = \infty$; therefore, $\lim_{R\to 0^-}\ln|R| - \lim_{S\to 0^+}\ln S$ does not exist (is that clear, by the way?)

(d) The approach in point (a) tells us that the «area under the graph» and the «area over the graph» are the same, equal to, say, A; however, for improper integrals, they may both be infinite. Then the integral would be $\infty - \infty$, which does not exist. But if the areas are finite, then the integral is $A - A = 0$. So the method can tell us that *IF the integral converges, then it is zero*. It does however tell you nothing about the convergence.

If you preferred the slightly more formal proof in point (a), then you might observe that everything up to and including relation (*) holds for improper integrals too. However (assuming $k > 0$), we have:

$$\begin{aligned}
&= \int_0^k f(x)dx + \int_R^0 (-f(y))\,(-dy) \\
&= \lim_{S\to 0^+}\int_S^k f(x)dx + \lim_{R\to 0^-}\int_k^R f(y)\,dy \\
&= F(k) - \lim_{S\to 0^+} F(S) + \lim_{R\to 0^-} F(R) - F(k) \\
&= F(0^+) - F(0^-).
\end{aligned}$$

By symmetry, we have $F(0^+) = F(0^-)$, provided that they exist (either both or none do!), so if $F(0)$ exists and is finite, the integral converges and is zero. But again, the calculations in point (a) – up to and including (*) – says nothing about whether the improper integral converges or not.

Comments problem 5: I did expect a few strange answers to this, and I got more than a few. Unfortunately (c) was not answered too well, and problems like point (c) could show up in an exam.
Point (a), turned out well for most of you. But please be aware of the difference between «definite integral» and «area»: An area is a positive number, but the integral is a positive or negative number which has positive contribution from the positive part of the function (area under the graph and over the x-axis) and negative contribution from the area under the axis and over the graph.
Point (b): This is something I would be reluctant to give at an exam. Many of you were on target, pointing out that the integrand is discontinuous at zero. It turned out an unexpected confusion on the value of $(-1)^{2/3}$.
Point (c): This was not too good: together with point (a), this was the ss-olve"question in Problem 5, which is something more suitable for an exam problem set. Quite a few of you were on target in (b), only to happily cancel the antiderivative limits in (c) without checking – if that approach were valid, then my calculations in (b) would have been valid too!
Point (d): hard and should not be given at an exam. You need to catch the essence of what you were asked to do in (a)–(c), and generalize it. Besides it draws so heavily on your answers to those points being correct in order to do anything at all, and you carried all sorts of mistakes with you from previous points (including those I had not at all foreseen).