# Note: Solution to compulsory term paper no. 2 in ECON3120/4120 Mathematics 2

The problem set was to a large extent taken from last spring's exam problem set, which is considered to be a bit too easy. And besides, I had done problem 3 in a lecture (thanks to the student who reminded me). So your grades may have come out a bit optimistic, as – just like last time, but with opposite sign this time – I have not done anything to compensate for the problem set being slightly off the average difficulty. The grade distribution here was 13 A's, 12 B's, 5 C's and *no* DEF.

Upon grading your papers, I weighted the problems as that exam, with the modifications that 4c was weighted down to half its original score (since you weren't asked to do the latter part of it) and the weight of problem 5 is the sum of 4d (which was deleted for this term paper) and half 4c. This gives the problems the following weights in percent points: 1: **10+12**, 2: **18**, 3: **8+12**, 4: **10+6+8**, 5: **16**.

The solution that was made for last spring's exam problem set is attached. From your papers I have made a few additional comments:

## Problem 1

(a) Most of you understand cofactor expansion, though I am obviously not the only one who gets signs wrong sometimes.

(b) For the case $a = 2$, the two last equations say the same. That is however not sufficient to conclude that there are many solutions; you must also show that the first equation does not contradict the two others. Consider the following counterexample:

$$\begin{aligned} 3x + 3y + 3z &= 2 \\ x + \phantom{3}y + \phantom{3}z &= 1 \\ x + \phantom{3}y + \phantom{3}z &= 1 \end{aligned} \qquad \text{(eq. 1b)}$$

Evidently the last two equations say the same. But the first says something different.

There is a way to check this: deleting the third line from the equation set (the one in problem 1 (b)) and the third line from $\mathbf{A}_2$, we are left with the coefficient matrix $\mathbf{B} = \begin{pmatrix} 3 & 2 & -4 \\ 1 & 1 & 1 \end{pmatrix}$. This one has three $2 \times 2$ submatrices, and if *at least one* of these has nonzero determinant, then there exists a solution, where the corresponding two variables are uniquely determined by the third one. For example, consider the matrix $\mathbf{C}_1$ obtained by deleting the first column; $\begin{vmatrix} 2 & -4 \\ 1 & 1 \end{vmatrix} \neq 0$ so we can choose $x = s$ freely and – given $s$ – there will be one and only one solution for $(y, z)$. Hence there is one degree of freedom. Similarly we could have considered $\mathbf{C}_2$ obtained by deleting the second column (would give $y = t$ free and $(x, z)$ uniquely in terms of $t$) and $\mathbf{C}_3$ obtained by deleting the third column. If at least one of $|\mathbf{C}_1|$, $|\mathbf{C}_2|$, $|\mathbf{C}_3|$ is nonzero, then we have one degree of freedom. If all three are zero, then we *either* have more than one degree of freedom *or* no solution – consider the example (eq. 1b) above where *all* $2 \times 2$ subdeterminants are zero, and the right hand side determines existence.

But bottom line: *you cannot, logically, find one superfluous line and conclude «many solutions», you must check that there aren't other inconsistencies.*

I have only reduced your score by one point for this error; this based on 12 points for 1b from 4 points for each of (i), (ii) and (iii), and this way you got to the right answer on (ii) by an insufficient argument ($\rightsquigarrow 3/4$ score). In retrospect, I should maybe have been a bit stricter.

(b cont'd) Also, you cannot use Cramer's rule and then simplify off $(a - 2)$ from all determinants; that is like inferring $x = b$ from $0x = 0b$.

**Problem 2** Quite a few of used wrong sign in the integrating factor $\rightsquigarrow$ 2 points reduction, based on weights of 12 points for the general solution and 6 points for deducing the particular from that one.

**Problem 3** A bit too late I was reminded that I had given *precisely* this problem in a lecture. Oh well. That won't repeat itself for the exam, I promise.

(b) You were asked for a *general* expression – so don't insert the point.

**Problem 4**

(a) Most did well on this.

(b)
- Please insert for $u'_x$ and $u'_y$ – one point (of six) reduced for not doing this.
- Remember the $u = K$ condition! Well: For those of you who rather than using $u = K$ directly as a constraint, used $Ax + \frac{1}{2}y^2 - K - \ln K$, I have accepted it – although I think that $u = K$ is definitely closer to what was asked for.

**Problem 5** A few general comments here: in some of the problems ((a), (g), (h)), you might need to establish the order of the matrix. You might want to know that if $\mathbf{A}^k$ is defined for $k > 1$, then $\mathbf{A}$ is necessarily square.

(a) If $\mathbf{AB}$ has an inverse, then $\mathbf{A}$ has an inverse. *(Hint: When is $\mathbf{AB}$ defined?)*
**False.** $\mathbf{A}$ need not be square. (However, if we knew that it is, then the assertion would have been true.)

(b) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$, then $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$.
**True.** See box EMEA p. 568 (LA p. 59). (Holds if $\mathbf{A}$ and $\mathbf{B}$ are of same order, not necessarily square!)
**Comment:** You cannot make proofs by example! That is logically flawed. (However, you can do counterproofs by counterexamples, like in point (c).)

(c) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$, then $|(\mathbf{A} + \mathbf{B})'| = |\mathbf{A}'| + |\mathbf{B}'|$. *(Hint: Try $\mathbf{A} = \mathbf{B}$.)*
**False.** The hint $\mathbf{A} = \mathbf{B}$ gives a counterexample: we get $|2\mathbf{A}|$ which (see box EMEA p. 602 (LA p. 98)) is equal to $2^n|\mathbf{A}|$, violating the assertion for any $n = 2, 3, \dots$.

(d) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$, then $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$. *(Hint: For (d), (e) and (f), calculate the difference between the LHS and the RHS. Is it always zero?)*
**False.** $(\mathbf{A}+\mathbf{B})^2$ is equal to $(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{A}(\mathbf{A}+\mathbf{B})+\mathbf{B}(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$, and if we subtract $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ we get $\mathbf{BA} - \mathbf{AB}$ which is *not* zero except in special cases.

(e) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ and both symmetric, then $(\mathbf{A}+\mathbf{B})'(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.
**False.** The same reason as in (d).
**Comment:** I am not sure that I have mentioned in the lectures, but a product of symmetric matrices is not symmetric except in special cases. At least, now you know.

(f) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ and both symmetric, then $(\mathbf{A}+\mathbf{B})'(\mathbf{A}+\mathbf{B}) = (\mathbf{A}+\mathbf{B})^2$ – even when $(\mathbf{A}+\mathbf{B})$ is singular.
**True.** You may calculate as the hint, or you may use (b) for a direct argument; by symmetry, the prime may be removed.

(g) If $\mathbf{A}^k = \mathbf{I}_n$ (for some $k$), then $|\mathbf{A}| = 1$ or $-1$, and only $1$ is possible if $k$ is an odd number.
**True.** We are told that $\mathbf{A}^k$ exists, so $\mathbf{A}$ has to be square, and the determinant is defined – and it solves the equation $d^k = 1$ (see box EMEA p. 602 (LA p. 98)).
**Comment:** Surprisingly many wrong answers on this one. Many of you seem to think that if $\mathbf{A}^k = \mathbf{I}$, then $\mathbf{A}$ has to be diagonal – or even equal to $\mathbf{I}$ or maybe $-\mathbf{I}$. That is incorrect: it is easy to show (by induction) that

$$\begin{pmatrix} 1 & r \\ 0 & a \end{pmatrix}^k = \begin{pmatrix} 1 & rs(a) \\ 0 & a^k \end{pmatrix}, \quad \text{where } s(a) = 1 + a + a^2 + \cdots + a^{k-1}.$$

So for $a = -1$ and $k$ even, the matrix power is equal to the identity.

(h) If $\mathbf{AB} = \mathbf{I}_n$ and $\mathbf{SAAB} = \mathbf{I}_n$, then we can conclude that $\mathbf{S} = \mathbf{B}$ without making the additional assumption that $\mathbf{A}$ is square. *(Hint: Maybe $\mathbf{A}$ will be square automatically?)*
**Comment:** A lot of you went straight from $\mathbf{AB} = \mathbf{I}_n$ to claiming that $\mathbf{B} = \mathbf{A}^{-1}$. That is true only when $\mathbf{A}$ is square: consider the counterexample $\mathbf{A} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right) = \mathbf{B}'$; then $\mathbf{AB} = \mathbf{I}_2$ but $\mathbf{BA}$ is not defined. The definition of an inverse requires it to be both a «left-inverse» and a «right-inverse» (EMEA p. 610, LA p. 109); however, if $\mathbf{A}$ is square it suffices to check that $\mathbf{B}$ is either a left inverse or a right inverse. This is why it is crucial to check that $\mathbf{A}$ is square – which in this case holds because there is an $\mathbf{AA}$ term defined.
**True.** I'll give two different arguments. The first one is a bit stepwise: Yes $\mathbf{A}$ is square – it has to be for $\mathbf{AA}$ to be defined. The number of rows – and hence the number of columns – is $n$ since $\mathbf{AB}$ has $n$ rows. Arguing this way, we see that all matrices are $n \times n$. Since $\mathbf{B}$ is the right-inverse of $\mathbf{A}$ and $\mathbf{SA} = \mathbf{SAAB} = \mathbf{I}_n$, then $\mathbf{S}$ the left-inverse of $\mathbf{A}$. But there is only one inverse, and $\mathbf{S} = \mathbf{B}$.
A more elegant argument is: We have $\mathbf{AB} = \mathbf{I}_n$, so $\mathbf{I}_n = \mathbf{SAAB} = \mathbf{SAI}_n = \mathbf{SA}$. Now we are allowed to post-multiply both sides of the equation $\mathbf{I}_n = \mathbf{SA}$ by $\mathbf{B}$ to get $\mathbf{SAB} = \mathbf{B}$, where the left hand side simplifies to $\mathbf{S}$.

(i) If the price vector $\mathbf{p}$, the initial endowment $\mathbf{a}$, and the post-trade endowment $\mathbf{x}$ are all $n$-vectors, with all prices $p_i > 0$ and $\mathbf{a} \neq \mathbf{0}$, then the equation $\mathbf{p} \cdot (\mathbf{x} - \mathbf{a}) = 0$ in the unknown $\mathbf{x}$, has $n-1$ degrees of freedom.
**True.** Choose all components of $\mathbf{x}$ except one – say, number $i$; since $p_i \neq 0$, you can then solve for that one uniquely: $x_i = a_i - \sum_{j \neq i}(x_j - a_j)p_j/p_i$.
**Comment:** After all this theory on the solutions of linear equations, some of you still seem to think that the counting rule is logically valid (*sigh*). Of course it isn't. If all $p_i$ were zero, we would have $\mathbf{0} = \mathbf{0}$ regardless of $\mathbf{x}$, that is, $n$ degrees of freedom.
**Comment:** A few of you appealed to Walras' law here. Walras' law says that the sum of excess demand vanishes. However, this is logically flawed – it refers to something

somewhat different, namely aggregates over agents, and is deduced by assuming that everyone obeys their budget constraints. But like the way of thinking a bit, so you have received a slight score.

(j) If $\mathbf{B} = (b_{ij})_{n\times n}$, where all $b_{ij} = 1$, then the equation $(\mathbf{B} - n\mathbf{I}_n)\mathbf{x} = \mathbf{0}$ has precisely two solutions, namely the null vector and the vector of only ones.
**False.** A linear equation system *never* has precisely two solutions.
**Comment:** Yes both of the given vectors solve the equation, but then there's the p-word: «precisely».

(k) If $\mathbf{A} = (a_{ij})$ is a diagonal matrix, and the equation $\mathbf{Ax} = \mathbf{b}$ has a solution, then the number of degrees of freedom is equal to the number of zeroes on the main diagonal. *(Hint: If there is a zero on the main diagonal, what can you do about that line/column?)*
**True.** Since the matrix is diagonal, all non-zero entries on the diagonal determine the respective variables uniquely: $\mathbf{b} = \mathbf{Ax} = (a_{11}x_1, \ldots, a_{nn}x_n)'$ so $x_i = b_n/a_{ii}$ if $a_{ii} \neq 0$. If however $(a_{ii}) = 0$ for some $i$, the corresponding $i$th says $0x_i = b_i$. (Since we are given that the problem has a solution, then necessarily $b_i = 0$.) This means that precisely these $x_i$ – all of them, and none more – can be chosen freely.
**Comment:** A few of you seem to think that the other variables must be influenced by these free variables in some way in order to say they contribute to the number of degrees of freedom. That is not correct; the equation system $x = 1,\ 0y = 0,\ 0z = 0$ has two degrees of freedom.