ECON3120/4120: **note on differentials and equation systems**

This note considers the problem of finding partial derivatives from functions given implicitly by equation systems. At the end, there is a specific equation set which will be treated in the lecture next Monday (2007-09-03).

Suppose we are given an equation system

$$\begin{aligned} f(x,y,z,u,v) &= 0 \\ g(x,y,z,u,v) &= 0 \end{aligned} \tag{1}$$

where f and g are continuously differentiable functions[1]. Assume that we know that the equation system defines u and v as differentiable functions $u(x,y,z)$ and $v(x,y,z)$ in a suitable domain (phrased e.g. as near a given point). This note gives a guide to finding the partial derivatives of u and v.

To differentiate the system (find the differential): use the definition (safe) or find the differentials term by term using the similar rules as for derivatives (e.g. $de^{v-u} = e^{v-u}d(v-u)$ etc. – possibly more efficient).

To find the partial derivatives $u'_x, u'_y, u'_z, v'_x, v'_y, v'_z$, solve for du and dv and pick the dx term, dy term or dz term as appropriate. Proceed as follows (simplifications might be possible, depending on the problem):

- The differentials define two equations. Consider them an equation system to be solved for du and dv, where everything else (x, y, u, v, dx, dy and dz) is treated as constants. It might be helpful to re-write the differentiated system by gathering the du and dv terms on the left hand sides and everything else on the right hand sides:

$$\begin{aligned} Adu + Bdv &= C \\ Ddu + Edv &= F \end{aligned} \tag{2}$$

- Now solve this for (du, dv) as usual (e.g. eliminate du from one equation).[2]
- Write out the right hand sides by gathering dx terms and dy terms separately:

$$\begin{aligned} du &= Gdx + Hdy + Kdz \\ dv &= Idx + Jdy + Ldz, \end{aligned} \tag{3}$$

[1]We have this far not considered z-dependence, but you should be able to verify from this note that the approach is the same.

[2]Unless $AE = BD$ – which is the case where the conditions granting u, v as *functions* do fail – then du, dv will be given as $du = (EC - BF)/(AE - BD)$ and $dv = (AF - CD)/(AE - BD)$. Later in this course you will learn that this solution is the inverse of the matrix $\begin{bmatrix} A & B \\ D & E \end{bmatrix}$, matrix-multiplied with the vector $\begin{bmatrix} C \\ F \end{bmatrix}$.

where the coefficients[3] $G, \dots, L$ may depend on both x, y, z, u and v. These coefficients are the partial derivatives. For example, $u'_y = H$.

If you are asked to find the approximate change in u and v from given increments in (x, y, z) from a point P satisfying the original equation system(4), then you proceed as follows:

- Consider (3). Insert in $G, \dots, L$ the coordinates of P for (x, y, z, u, v).
- Insert the increments in x, y and z for dx, dy and dz – and remember the signs!
- Calculate du and dv.

On Monday's lecture, we will consider the following example, slightly modified from Example 2 p. 455 in EMEA:

$$\begin{aligned} u^2 + v &= xy + z \\ uv &= -x^2 + y^2. \end{aligned} \tag{4}$$

[3] I chose to keep the same name of the terms as in the lecture, hence not alphabetical order for dz