ECON3120/4120 – Mathematics 2, fall term 09: **Solutions for seminar 4, Sep. 23**

You requested solutions for exam problems 43 and 97.

### Problem 43

$$\begin{aligned}\frac{d}{dN}\left(Ng\left(\frac{\phi(N)}{N}\right)\right) &= g\left(\frac{\phi(N)}{N}\right)+N\frac{d}{dN}g\left(\frac{\phi(N)}{N}\right)\\ &= g\left(\frac{\phi(N)}{N}\right)+N\left(g'\left(\frac{\phi(N)}{N}\right)\frac{\phi'(N)N-\phi(N)}{N^2}\right)\\ &= g\left(\frac{\phi(N)}{N}\right)+g'\left(\frac{\phi(N)}{N}\right)\phi'(N)-g'\left(\frac{\phi(N)}{N}\right)\frac{\phi(N)}{N}\end{aligned}$$

The next one is a bugger. To simplify use the notation:

$$u=\frac{\phi(N)}{N}\qquad\text{and}\qquad h(N)=\frac{\phi'(N)N-\phi(N)}{N^2}$$

to obtain

$$\begin{aligned}&\frac{d^2}{dN^2}\left(Ng\left(\frac{\phi(N)}{N}\right)\right)\\ &=\frac{d}{dN}\left(g\left(\frac{\phi(N)}{N}\right)+g'\left(\frac{\phi(N)}{N}\right)\phi'(N)-g'\left(\frac{\phi(N)}{N}\right)\frac{\phi(N)}{N}\right)\\ &=g'(u)h(N)+g''(u)h(N)\phi'(N)+g'(u)\phi''(N)-\big(g''(u)h(N)u+g'(u)h(N)\big)\\ &=g''(u)h(N)\left(\phi'(N)-u\right)+g'(u)\phi''(N)\\ &=g''(u)\frac{\phi'(N)N-\phi(N)}{N^2}\left(\phi'(N)-\frac{\phi(N)}{N}\right)+g'(u)\phi''(N)\\ &=\frac{g''(u)}{N}\left(\phi'(N)-\frac{\phi(N)}{N}\right)\left(\phi'(N)-\frac{\phi(N)}{N}\right)+g'\left(\frac{\phi(N)}{N}\right)\phi''(N)\\ &=\frac{g''(u)}{N}\left(\phi'(N)-u\right)^2+g'(u)\phi''(N)\end{aligned}$$

**Problem 97** We have $\varphi(x)=\ln\frac{x+1}{x+2}$, defined for $x\geq 0$.
We can re-write into $\varphi(x)=\ln(1-\frac{1}{x+2})$. Now $1/(x+2)$ is strictly decreasing, so $1-\frac{1}{x+2}$ is strictly increasing, and since ln is strictly increasing, then so is $\varphi$.

(a) We have $\varphi(0)=\ln\frac{1}{2}=-\ln 2$, while $\varphi(x)\nearrow\ln 1$ as $x\to\infty$ (in a strictly increasing manner). So the range of $\varphi$ is the interval $\underline{\underline{[-\ln 2, 0)}}$.

(b) The inverse is defined on the range of $\varphi$, which by (a) is $\underline{\underline{[-\ln 2, 0)}}$. To find $\varphi^{-1}$, we solve

$x = \varphi(y)$ for $y$:

$$\begin{aligned} x = \ln(1 - \frac{1}{y+2}) \qquad &\iff \\ e^x = 1 - \frac{1}{y+2} \qquad &\iff \\ \frac{1}{y+2} = 1 - e^x \qquad &\iff \\ y = \underline{\underline{(1-e^x)^{-1} - 2}}. \end{aligned}$$

(note that the given solution can be obtained by rewriting into a common fraction.)

(c) Denote $\psi = \varphi'$ and and approach as above. First, $\psi(x) = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+1)(x+2)}$. (This is clearly strictly decreasing for $x \geq 0$, so an inverse exists, but we can take the latter for granted from the problem.)
$\psi^{-1}$ is defined on the range of $\psi$. Now $\psi$ ranges between $\psi(0) = \frac{1}{2}$ (and $\frac{1}{2}$ is in the range!) and the limit as $x \to \infty$, which is zero (but zero is not attained by $\psi$). So the domain of $\psi^{-1}$ equals $\underline{\underline{(0, \frac{1}{2}]}}$ (note the half-openness).

To find $\psi^{-1}$, we have

$$x = \frac{1}{(y+1)(y+2)}.$$

Solving for $y$, we get

$$y^2 + 3y + 2 - \tfrac{1}{x} = 0,$$

which has solutions

$$y = \tfrac{1}{2}\big[-3 \pm \sqrt{9 - 8 + \tfrac{4}{x}}\big]$$

Since the domain of $\psi$ contains only nonnegatives, we must have $y \geq 0$ too (as $x$ ranges $(0, \frac{1}{2}]$), and for this to be possible, then «$\pm$» must be «$+$». We get

$$(\varphi')^{-1}(x) = \underline{\underline{\tfrac{1}{2}\big[-3 + \sqrt{1 + \tfrac{4}{x}}\big]}}$$