Solutions to the cancelled seminar September 30

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a)

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{s}{12} & \frac{t}{12} & \frac{1}{4} \\ \frac{7}{12} & -\frac{2}{3} & \frac{1}{4} \\ \frac{1}{12} & \frac{t}{12} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{s}{12}+\frac{17}{12} & \frac{t}{3}-\frac{4}{3} & 0 \\ \frac{s}{6}+\frac{5}{6} & \frac{5t}{12}-\frac{2}{3} & 0 \\ \frac{s}{4}+\frac{5}{4} & \frac{t}{3}-\frac{4}{3} & 1 \end{pmatrix} = \begin{pmatrix} \frac{s+17}{12} & \frac{t-4}{3} & 0 \\ \frac{s+5}{6} & \frac{1}{12}(5t-8) & 0 \\ \frac{s+5}{4} & \frac{t-4}{3} & 1 \end{pmatrix}$$

Note that if **A** and **B** are inverse of each other then **AB = I**$_n$. That is:

$$\begin{bmatrix} \frac{s+17}{12} & \frac{t-4}{3} & 0 \\ \frac{s+5}{6} & \frac{1}{12}(5t-8) & 0 \\ \frac{s+5}{4} & \frac{t-4}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using this we can calculate e.g. that $\frac{s+17}{12} = 1 \Rightarrow s = -5$ and $\frac{t-4}{3} = 0 \Rightarrow t = 4$.

However, we need to check all the entries where *s* and *t* occur. They all give the same answer so we have solved the problem.

b) Remeber that matrices must be multiplied from the same side.

$$\begin{aligned} &\mathbf{BX} = 2\mathbf{X} + \mathbf{C} \\ &\mathbf{BX} - 2\mathbf{I}_3\mathbf{X} = \mathbf{C} \\ &\left(\mathbf{B} - 2\mathbf{I}_3\right)\mathbf{X} = \mathbf{C} \\ &\mathbf{X} = \left(\mathbf{B} - 2\mathbf{I}_3\right)^{-1}\mathbf{C} \end{aligned}$$

Calculating the inverse is tedious, but note that: $\left(\mathbf{B} - 2\mathbf{I}_3\right) = \mathbf{A}$. This implies that $\left(\mathbf{B} - 2\mathbf{I}_3\right)^{-1} = \mathbf{T}$ when *s* = –5 and *t* = 4. **T** then becomes

$$\begin{pmatrix} -\frac{5}{1} & \frac{1}{23} & \frac{1}{4} \\ \frac{7}{1} & -\frac{2}{23} & \frac{1}{4} \\ \frac{1}{1} & \frac{1}{23} & -\frac{1}{4} \end{pmatrix}$$

Right-multiplying **T** with **C** gives

$$\mathbf{X} = \mathbf{TC} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{6} \\ \frac{1}{2} & 3 & -3 & \frac{1}{6} \\ \frac{1}{2} & -1 & 2 & \frac{1}{6} \end{pmatrix}$$

c)

$$\begin{aligned} \mathbf{D}^2 &= 2\mathbf{D} + 3\mathbf{I}_n \\ \mathbf{D}^3 &= 2\mathbf{D}^2 + 3\mathbf{D} \\ \mathbf{D}^3 &= 2\left(2\mathbf{D} + 3\mathbf{I}_n\right) + 3\mathbf{D} \\ \mathbf{D}^3 &= 7\mathbf{D} + 6\mathbf{I}_n \end{aligned}$$

To find $\mathbf{D}^6$ try squaring $\mathbf{D}^3$.

$$\begin{aligned} &\mathbf{D}^6 = \mathbf{D}^3\mathbf{D}^3 = \left(7\mathbf{D} + 6\mathbf{I}_n\right)\left(7\mathbf{D} + 6\mathbf{I}_n\right) \\ &= 49\mathbf{D}^2 + 2 \times 42\mathbf{D} + 36\mathbf{I}_n \\ &= 49\left(2\mathbf{D} + 3\mathbf{I}_n\right) + 84\mathbf{D} + 36\mathbf{I}_n \\ &= 182\mathbf{D} + 183\mathbf{I}_n \end{aligned}$$

To find $\mathbf{D}^{-1}$, do as follows:

$$\begin{aligned} &\mathbf{D}^2 = 2\mathbf{D} + 3\mathbf{I}_n \qquad\qquad | \times\mathbf{D}^{-1} \\ &\mathbf{D} = 2\mathbf{I}_n + 3\mathbf{D}^{-1} \\ &\mathbf{D}^{-1} = \frac{1}{3}\mathbf{D} - \frac{2}{3}\mathbf{I}_n \end{aligned}$$

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a) Often when integrating an expression containing a square root it pays to use a substitution of the form $u = \sqrt{t}$. Then $du = \left(2\sqrt{t}\right)^{-1} dt$. We have that:

$$\int_1^4 e^{-\sqrt{t}}dt = \int_1^4 e^{-\sqrt{t}}\,\frac{2\sqrt{t}}{2\sqrt{t}}dt = \int_1^4 2e^{-u}udt$$

The last integral may be solved by integration by parts

$$g' = 2e^{-u} \to g = -2e^{-u}$$
$$f = u \to f' = 1$$

We may then write:

$$\begin{aligned} F\left(t\right) &= \int 2e^{-u}u = -2e^{-u}u - \int -2e^{-u}du \\ &= -2e^{-u}u - 2e^{-u} + C \\ &= -2e^{-u}\left(u+1\right) + C \\ &= -2e^{-\sqrt{t}}\left(\sqrt{t}+1\right) + C \end{aligned}$$

Computing the definite integral gives

$$\begin{aligned} F\left(4\right) - F\left(1\right) &= -2e^{-2}\left(2+1\right) - \left(-2e^{-1}\left(1+1\right)\right) \\ &= -6e^{-2} + 4e^{-1} \end{aligned}$$

b)

Several ways of doing this. Here is one. Start with the first equation.

$$\begin{aligned} &\mathbf{AX} + \mathbf{Y} = \mathbf{C} \qquad \mid \text{ left-multiply by } \mathbf{A}^{-1} \\ &\mathbf{X} + \mathbf{A}^{-1}\mathbf{Y} = \mathbf{A}^{-1}\mathbf{C} \\ &\mathbf{A}^{-1}\mathbf{Y} = \mathbf{A}^{-1}\mathbf{C} - \mathbf{X} \end{aligned}$$

Insert for $\mathbf{A}^{-1}\mathbf{Y}$ into the second equation and get:

$$\begin{aligned} &\mathbf{X} + 2\mathbf{A}^{-1}\mathbf{Y} = \mathbf{D} \\ &\mathbf{X} + 2\left(\mathbf{A}^{-1}\mathbf{C} - \mathbf{X}\right) = \mathbf{D} \\ &\mathbf{X} - 2\mathbf{X}\left(\mathbf{A}^{-1}\mathbf{C} - \mathbf{X}\right) = \mathbf{D} \\ &\mathbf{X} = \mathbf{2A}^{-1}\mathbf{C} - \mathbf{D} \end{aligned}$$

Return to the first equation.

$$\begin{aligned} &\mathbf{AX} + \mathbf{Y} = \mathbf{C} \\ &\mathbf{Y} = \mathbf{C} - \mathbf{AX} \\ &\mathbf{Y} = \mathbf{C} - \mathbf{A}\left(2\mathbf{A}^{-1}\mathbf{C} - \mathbf{D}\right) \\ &\mathbf{Y} = \mathbf{C} - 2\mathbf{C} + \mathbf{AD} \\ &\mathbf{Y} = \mathbf{AD} - \mathbf{C} \end{aligned}$$

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a)

$$\det \mathbf{A}_t = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & t \end{vmatrix}$$
$$= 1\times 1\times t + 1\times 1\times 1 + 3\times(-1)\times(-1) - 1\times 1\times 3 - (-1)\times 1\times t - 1\times(-1)\times 1$$
$$= t + 1 + 3 - 3 + t + 1 = 2t + 2$$

b)

$$\left[\begin{array}{cccc} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 3 & 1 & -1 & 4 \end{array}\right] \begin{array}{cc} -1 & -3 \\ \hookleftarrow & \downarrow \\ & \hookleftarrow \end{array} \sim \left[\begin{array}{cccc} 1 & -1 & 1 & 2 \\ 0 & 2 & -2 & -1 \\ 0 & 4 & -4 & -2 \end{array}\right] \begin{array}{c} \\ \times\frac{1}{2} \\ \times\frac{1}{4} \end{array}$$

$$\sim \left[\begin{array}{cccc} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} \end{array}\right] \begin{array}{c} \\ -1 \\ \hookleftarrow \end{array} \sim \left[\begin{array}{cccc} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can here choose x_3 freely. Let $x_3 = s$. Then we have that $x_2 = x_3 - ½ = s - ½$ and $x_1 = x_2 - x_3 + 2 = s - ½ - s + 2 = 3/2$.

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a) The answer is $3 + 4a^2$. (Do cofactor expansion.)

b)

$$\begin{bmatrix} 1 & 1 & a & a & a \\ 2 & 1 & -a^2 & 2a & 1 \\ 4 & 3 & a^2 & 4a^2 & 1 \end{bmatrix} \begin{matrix} -2 & -4 \\ \hookleftarrow & \downarrow \\ & \hookleftarrow \end{matrix} \sim \begin{bmatrix} 1 & 1 & a & a & a \\ 0 & -1 & -2a-a^2 & 0 & -2a+1 \\ 0 & -1 & -4a+a^2 & -4a+4a^2 & -4a+1 \end{bmatrix} \begin{matrix} \\ \times(-1) \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & a & a & a \\ 0 & 1 & 2a+a^2 & 0 & 2a-1 \\ 0 & -1 & -4a+a^2 & -4a+4a^2 & -4a+1 \end{bmatrix} \begin{matrix} \\ 1 \\ \hookleftarrow \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & a & a & a \\ 0 & 1 & 2a+a^2 & 0 & 2a-1 \\ 0 & 0 & -2a+2a^2 & -4a+4a^2 & -2a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & a & a & a \\ 0 & 1 & 2a+a^2 & 0 & 2a-1 \\ 0 & 0 & 2a(a-1) & 4a(a-1) & -2a \end{bmatrix}$$

Now examine the case where $a = 0$. Then the last matrix becomes:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We only get to determine x and x. z and u can be chosen freely. And we have two degrees of freedom. If $a = 1$, we get:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The last line in this matrix implies a contradiction. $0x + 0y + 0z + 0u$ cannot be –2, so for a = 1 there is no solution. Finally, if $a \neq 0$ and $a \neq 1$, then the last line becomes an equation on the form: $2a(a-1)z + 2a(a-1)u = -2a$. If we fix a value of u, then we determine z. Going backwards we can then determine x and y. In this case the system has one degree of freedom.

c)

See the answers section. Note that you cannot solve this exercise by assuming that **A** has an inverse. You get the “right” answer, but the exercise does not state that the inverse exists. (Example: the formula works also if **A** = **0**.)

a) We get that:

$$f'(x) = 4x - \frac{1}{x},\ f'(x) = 0 \Rightarrow x = \frac{1}{2}$$
$$f'' = 4 + \frac{1}{x^2} > 0 \text{ implies that } x = \frac{1}{2} \text{ is a global min point.}$$

b)

Again, many ways to solve this. I start by noting that $f'(1) = 4 - \frac{1}{4} > 0$. Thus there are values of $x < 1$ where $f(x) < 0$. Further, $\lim_{x \to 0^+} f(x) = \infty$. So clearly there are values of $x > 0$ such that $f(x) > 0$. It follows from the continuity of $f(x)$ that there is at least one solution for the equation $f(x_1) = 0$ that lies in (0, 1).

c)

Note that $g(x) = 1/f(x)$. We then have that:

$$g'(x) = -\frac{f'(x)}{\left(f(x)\right)^2}$$

Clearly $g'(x) = 0 \Leftrightarrow f'(x) = 0 \Leftrightarrow x = \frac{1}{2}$ by a) (here we have used that $f\left(\frac{1}{2}\right) \neq 0$) so $x = ½$ is a stationary point. Computing the second derivative $g''(x)$ is messy (but possible), and the following is simpler: $x = ½$ is a (strict) minimum point for f, and since $1/f$ is locally strictly decreasing with respect to f, a strict local minimum for f is a strict local maximum for g. (Why? Because any nearby x-value yields slightly higher f-value, meaning we divide by slightly more.)
(Why does this argument not imply strict *global* max? Because $1/f$ is not decreasing around $f=0$. Indeed, we know from a) that f "crosses through" zero (twice, actually) and nearby these x-values, we will have g arbitrary large positive, and arbitrary large negative. Hence no global extrema.

d)
See answers section