Universitetet i Oslo / Økonomisk institutt / NCF

ECON3120/4120 – Mathematics 2, fall term 09: **Solutions for seminar 9, Nov. 4**

This note contains solutions for problems 59 and 73 which were incompletely covered at the seminar. The following solutions are kindly provided by Arne Strøm:

Exam problem 59

(a) Direct calculation yields

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix}$$

$$\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 30 & 5 & 1 \end{pmatrix}$$

$$\mathbf{I}_3 - \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -10 & -5 & 1 \end{pmatrix}$$

and, finally,

$$(\mathbf{I}_3 - \mathbf{A})(\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2) = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -10 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 30 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3.$$

(b) By the last result in (a),

$$(\mathbf{I}_3 - \mathbf{A})^{-1} = \mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 30 & 5 & 1 \end{pmatrix}.$$

(c) $(\mathbf{I}_n + a\mathbf{U})(\mathbf{I}_n + b\mathbf{U}) = \mathbf{I}_n + a\mathbf{U} + b\mathbf{U} + ab\mathbf{U}^2$. But

$$\mathbf{U}^2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \dots & n \end{pmatrix} = n\mathbf{U},$$

and consequently,

$$\mathbf{I}_n + a\mathbf{U} + b\mathbf{U} + ab\mathbf{U}^2 = \mathbf{I}_n + a\mathbf{U} + b\mathbf{U} + nab\mathbf{U} = \mathbf{I}_n + (a + b + nab)\mathbf{U}.$$

(d) Let us call the given matrix $\mathbf{D}$. It is easy to see that

$$\mathbf{D} = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \mathbf{I}_3 + 3\mathbf{U}.$$

Let b be an arbitrary number. From the result in (c), we get

$$(\mathbf{I}_3 + 3\mathbf{U})(\mathbf{I}_3 + b\mathbf{U}) = \mathbf{I}_3 + (3 + b + 3 \cdot 3b\mathbf{U}) = \mathbf{I}_3 + (3 + 10b)\mathbf{U}.$$

If we choose $b = -3/10$, then the last matrix expression above equals $\mathbf{I}_3$, and it follows that

$$\begin{aligned}\mathbf{D}^{-1} &= (\mathbf{I}_3 + 3\mathbf{U})^{-1} = \mathbf{I}_3 - (3/10)\mathbf{U} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3/10 & 3/10 & 3/10 \\ 3/10 & 3/10 & 3/10 \\ 3/10 & 3/10 & 3/10 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 7 & -3 & -3 \\ -3 & 7 & -3 \\ -3 & -3 & 7 \end{pmatrix}.\end{aligned}$$

Exam problem 73 With the Lagrangian

$$\mathcal{L}(x, y, z) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 + 4z^2 - 1) - \mu(x + 3y + 2z)$$

the necessary first-order conditions for maximum are

$$(\mathcal{L}'_1(x, y, z) =) \qquad 2x - 2\lambda x - \mu = 0 \tag{1}$$
$$(\mathcal{L}'_2(x, y, z) =) \qquad 2y - 2\lambda y - 3\mu = 0 \tag{2}$$
$$(\mathcal{L}'_3(x, y, z) =) \qquad 2z - 8\lambda z - 2\mu = 0 \tag{3}$$

together with the constraints

$$x^2 + y^2 + 4z^2 = 1 \tag{4}$$
$$x + 3y + 2z = 0 \tag{5}$$

Equation (1) gives

$$\mu = 2x - 2\lambda x = 2(1 - \lambda)x \tag{6}$$

We substitute this expression for μ in (2), and get

$$2(1 - \lambda)y - 6(1 - \lambda)x = 0 \iff 2(1 - \lambda)(y - 3x) = 0.$$

Hence, $\lambda = 1$ or $y = 3x$ (or both).

A. Suppose $\underline{\lambda = 1}$. Then (6) gives $\underline{\mu = 0}$, and (3) gives $2z - 8z = 0$, that is, $z = 0$. It then follows from (5) that $x = -3y$, and equation (4) gives $9y^2 + y^2 = 1$, so $y = \pm\sqrt{1/10} = \pm 1/\sqrt{10}$.
This leads to two solutions of the first-order equations:

$$(x_1, y_1, z_1) = \Big(-\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\Big), \qquad (x_2, y_2, z_2) = \Big(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, 0\Big).$$

B. Now assume that $\underline{\lambda \neq 1}$. Then $y = 3x$. Equation (5) gives $2z = -x - 3y = -10x$, so $z = -5x$. If we use this in (4), we get

$$x^2 + (3x)^2 + 4(-5x)^2 = 1 \iff x^2 + 9x^2 + 100x^2 = 1 \iff x = \pm\frac{1}{\sqrt{110}}.$$

This gives us the two points

$$
\begin{aligned}
(x_3, y_3, z_3) &= \Big(\frac{1}{\sqrt{110}}, \frac{3}{\sqrt{110}}, -\frac{5}{\sqrt{110}}\Big), \\
(x_4, y_4, z_4) &= \Big(-\frac{1}{\sqrt{110}}, \frac{-3}{\sqrt{110}}, \frac{5}{\sqrt{110}}\Big).
\end{aligned}
$$

The corresponding values of λ and μ can be found as follows: With $z = -5x$, equations (1) and (3) above become

$$
\begin{aligned}
2x - 2\lambda x - \mu &= 0 \\
-10x + 40\lambda x - 2\mu &= 0
\end{aligned}
\quad \Longleftrightarrow \quad
\begin{aligned}
2x\lambda + \mu &= 2x \\
20x\lambda - \mu &= 5x
\end{aligned}
$$

If we consider the last system as a linear equation system with λ and μ as the unknowns, it is easy to show that

$$
\underline{\lambda = \frac{7}{22}} \qquad \text{and} \qquad \underline{\mu = \frac{15}{11}x = \pm\frac{15}{11\sqrt{110}}}\,.
$$

Calculating the value of $f(x, y, z) = x^2 + y^2 + x^2$ at each of the four points that we have found, we get

$$
\begin{aligned}
f(x_1, y_1, z_1) = f(x_2, y_2, z_2) &= \frac{9}{10} + \frac{1}{10} + 0 = 1, \\
f(x_3, y_3, z_3) = f(x_4, y_4, z_4) &= \frac{1}{110} + \frac{9}{110} + \frac{25}{110} = \frac{35}{110} = \frac{7}{22}\,.
\end{aligned}
$$

This shows that (x_1, y_1, z_1) and (x_2, y_2, z_2) are global maximum points for f under the given constraints, *provided there is a maximum*.

How can we be sure that there is a maximum? The constraints determine a close and bounded set, and f is continuous, so the extreme value theorem ensures that f does attain both a maximum and a minimum under these constraints. It then also follows that the points (x_3, y_3, z_3) and (x_4, y_4, z_4) are minimum points.

Comment 1: Since we know that f really attains both a maximum and a minimum, it is not strictly necessary to determine the Lagrange multipliers when we look for global extreme points. All we need is to be sure that we have found all points that satisfy the Lagrange conditions. If we happen to include a few extra points, it does no harm, as these points will be exposed when we calculate the function values at all the candidate points. Think about it!

Comment 2: After all this it is almost embarrassing to point out that the whole thing would have been much easier if we had taken another look at the functions in the problem. It follows from constraint (4) that $x^2 + y^2 = 1 - 4z^2$, so the maximand, $f(x, y, z)$, equals

$$
x^2 + y^2 + z^2 = (1 - 4z^2) + z^2 = 1 - 3z^2
$$

throughout the admissible set. Hence, f certainly attains its maximum value at a point where $z = 0$. If we insert this value of z into (4) and (5), we find precisely the points (x_1, y_1, z_1) and (x_2, y_2, z_2) that we found above, and we know that these points must be maximum points, without having to worry about either the extreme value theorem or Lagrange's method. Oh well, that's life!