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# Answers to the examination problems in ECON 3120/4120, 26 May 2004

## Problem 1

(a) $f'(x) = -1 + \dfrac{1}{x-1} = \dfrac{2-x}{x-1}, \quad f''(x) = \dfrac{-1}{(x-1)^2}\,.$

(b) $f(x)$ is (strictly) increasing in $(1, 2]$ and (strictly) decreasing in $[2, \infty)$. The function has a (global) maximum at $x = 2$. There are no other extreme points, because $x = 2$ is the only point where $f'(x) = 0$.

(c) $\lim_{x\to 1^+} f(x) = -\infty$ because $4 - x \to 3$ and $\ln(x - 1) \to -\infty$. To find the limit as $x \to \infty$, note that we can write the function as

$$f(x) = xg(x), \quad \text{where} \quad g(x) = \frac{4}{x} - 1 + \frac{\ln(x-1)}{x}\,.$$

By L'Hôpital's rule, $\displaystyle\lim_{x\to\infty} \frac{\ln(x-1)}{x} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to\infty} \frac{1/(x-1)}{1} = 0$. Therefore, $g(x) \to -1$ and $f(x) = xg(x)$ tends to $-\infty$ as $x \to \infty$.

(d) $f(2) = 2 + \ln 1 = 2$ and $f(x) \to -\infty$ as $x \to 1^+$. Thus there exists an $x_1$ close to 1 with $f(x_1) < 0$. The intermediate value theorem (“skjæringssetningen”) tells us that the equation $f(x) = 0$ has at least one solution in the interval $(x_1, 2)$. Also, $f(x)$ is strictly increasing in $(1, 2]$. Hence, $f(x) = 0$ has a *unique* solution in the interval $(1, 2)$. In a similar way we see that $f(x) = 0$ also has a unique solution in $(2, \infty)$. It follows that the equation $f(x) = 0$ has exactly two roots.

Sketch a graph!

## Problem 2

(i) $\displaystyle\int (x^3 + 2x)^2\,dx = \int (x^6 + 4x^4 + 4x^2)\,dx = \tfrac{1}{7}x^7 + \tfrac{4}{5}x^5 + \tfrac{4}{3}x^3 + C.$

(ii) To find the integral $I = \displaystyle\int_0^{\sqrt{8}} \frac{x}{(1+x^2)^a}\,dx$ we try the substitution $1 + x^2 = u$, $2x\,dx = du$. Since $x = 0 \Rightarrow u = 1$ and $x = \sqrt{8} \Rightarrow u = 9$, we get

$$\begin{aligned} I &= \frac{1}{2}\int_1^9 \frac{du}{u^a} = \frac{1}{2}\int_1^9 u^{-a}\,du = \frac{1}{2}\Big|_1^9 \frac{1}{1-a}u^{1-a} \\ &= \frac{1}{2}\Big[\frac{1}{1-a}9^{1-a} - \frac{1}{1-a}\Big] = \frac{9^{1-a} - 1}{2(1-a)}\,. \end{aligned}$$



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**Problem 3**

(a) $\mathbf{A}'\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and $\mathbf{A}'\mathbf{A}$ is $|\mathbf{A}'\mathbf{A}| = 4 - 1 = 3$.

(b) $\mathbf{A}'\mathbf{A}$ has an inverse because $|\mathbf{A}'\mathbf{A}| \neq 0$. The inverse is

$$(\mathbf{A}'\mathbf{A})^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

**Problem 4**

(a) Differentiating with respect to $x$ yields

$$2(x^2 + y^2)(2x + 2yy') = a^2(2x - 2yy')$$

(we consider $y$ as a function of $x$), and therefore

$$y' = \frac{x}{y}\,\frac{a^2 - 2(x^2 + y^2)}{a^2 + 2(x^2 + y^2)}.$$

(b) At a point where the tangent is horizontal we must have $y' = 0$ and $x \neq 0$, so

$$x^2 + y^2 = \tfrac{1}{2}a^2. \tag{i}$$

We must also have

$$(x^2 + y^2)^2 = a^2(x^2 - y^2), \tag{ii}$$

because the point must lie on the lemniscate. From equation (i) we get $x^2 = \frac{1}{2}a^2 - y^2$, and then (ii) yields

$$\tfrac{1}{4}a^4 = a^2(\tfrac{1}{2}a^2 - 2y^2) \iff \tfrac{1}{4}a^2 = \tfrac{1}{2}a^2 - 2y^2 \iff y^2 = \tfrac{1}{8}a^2.$$

Hence $y = \pm\frac{1}{4}a\sqrt{2}$. Then $x^2 = \frac{1}{2}a^2 - y^2 = \frac{3}{8}a^2 = \frac{6}{16}a^2$, so $x = \pm\frac{1}{4}a\sqrt{6}$. It follows that the tangent to the curve is horizontal at the four points

$$\left(\pm\frac{a\sqrt{6}}{4}, \pm\frac{a\sqrt{2}}{4}\right),$$

**Problem 5**

With the Lagrangian $\mathcal{L}(x,y) = x + xy - \lambda(y + x^2e^y - 1)$, the necessary conditions for $(x, y)$ to solve the problem are

$$\partial\mathcal{L}/\partial x = 1 + y - 2\lambda x e^y = 0 \tag{1}$$

$$\partial\mathcal{L}/\partial y = x - \lambda - \lambda x^2 e^y = 0 \tag{2}$$

$$y + x^2 e^y = 1 \tag{3}$$

These equations are all satisfied at $(x_0, y_0) = (1, 0)$ if $\lambda = 1/2$.