University of Oslo
Department of Economics
A. Strøm

Answers to the examination problems in ECON3120/4120 Mathematics 2, 2 June 2009

Problem 1

(a) The partial derivatives are

$$f_1'x, y = 3e^{3x} + 3ye^x, \qquad f_2'(x, y) = 3e^x - 3y^2$$
$$f_{11}''(x, y) = 9e^{3x} + 3ye^x, \qquad f_{12}''(x, y) = 3e^x, \qquad f_{22}''(x, y) = -6y$$

(b) A point (x, y) is a stationary point for f if and only if

$$\begin{matrix} f_1'(x, y) = 0 \\ f_2'(x, y) = 0 \end{matrix} \iff \begin{matrix} 3e^x(e^{2x} + y) = 0 \\ 3(e^x - y^2) = 0 \end{matrix} \iff \begin{matrix} y = -e^{2x} \\ e^x = y^2 \end{matrix}$$

From the last pair of equations we get $e^x = e^{4x} \iff x = 4x$, which has the unique solution $x = 0$, and then $y = -e^{2x} = -1$. Thus, f has exactly one stationary point, namely $(0, -1)$.

To determine the nature of this stationary point we use the second-derivative test with $A = f_{11}''(0, -1) = 6$, $B = f_{12}''(0, -1) = 3$, $C = f_{22}''(0, -1) = 6$. We see that $A > 0$ and $AC - B^2 = 27 > 0$, and it follows that $(0, -1)$ is a local minimum point for f.

(*Comment:* It is not a global minimum point, because $f(0, y) = 1 + 3y - y^3$ tends to $-\infty$ as $y \to \infty$.)

(c) The equation $f(x, y) = 3$ determines y as a function of x in an open set around $(x_0, y_2) = (0, -2)$. The slope of the tangent to the curve at this point is

$$y' = -\frac{f_1'(0, -2)}{f_2(0, -2)} = -\frac{3e^0 - 6e^0}{3e^0 - 3(-2)^2} = -\frac{1}{3}\,.$$

The tangent is therefore given by the equation

$$y - (-2) = -\tfrac{1}{3}(x - 0) \iff y = -\tfrac{1}{3}x - 2.$$

M2xv09fasit 5.6.2009 774

Problem 2

(a) The derivative of f is $f'(x) = 2xe^x + x^2e^x = x(x+2)e^x$, which has the same sign as $x(x+2)$. It can be seen from a sign diagram that

$$\begin{aligned} f'(x) &> 0 \text{ if } x < -2 \\ f'(x) &< 0 \text{ if } -2 < x < 0 \\ f'(x) &> 0 \text{ if } x > 0 \end{aligned}$$

Since f is continuous everywhere, this implies that f is strictly increasing over $(-\infty, -2]$, strictly decreasing over $[-2, 0]$, and strictly increasing again over $[0, \infty)$.

Hence, f is one-to-one over $I_1 = (-\infty, -2)$, but not over $I_2 = (-\infty, 0)$ or $I_3 = (-2, \infty)$. (*Hint:* A sketch of the graph of f will help you see what happens.) It follows that f restricted to I_1 has an inverse. Over I_2 or I_3 the function does not have an inverse function.

(b) From the inverse function theorem (Theorem 7.3.1 in EMEA or Theorem 7.1.1 in MA I) we get

$$g'(f(x_0)) = \frac{1}{f'(x_0)} = \frac{1}{x_0(x_0+2)e^{x_0}}\,.$$

Problem 3

(a) Gaussian elimination yields

$$\begin{pmatrix} 1 & 1 & -3 & a \\ 1 & -3 & 4 & b \\ 3 & -1 & -2 & c \end{pmatrix} \begin{matrix} -1 & -3 \\ \longleftarrow & \\ \longleftarrow & \end{matrix} \sim \begin{pmatrix} 1 & 1-3 & a \\ 0 & -4 & 7 & b-a \\ 0 & -4 & 7 & c-3a \end{pmatrix} \begin{matrix} \\ -1 \\ \longleftarrow \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -3 & a \\ 0 & -4 & 7 & b-a \\ 0 & 0 & 0 & c-b-2a \end{pmatrix}$$

From the last matrix here it is clear that the system has solutions if and only if $c = b + 2a$.

(*Comment:* There is no need to carry the elimination process any further. It is also clear that if the system has solutions, then the solutions have 1 degree of freedom.)

(b) Matrix multiplication gives

$$\mathbf{AB} = \begin{pmatrix} 1 & t+9 & 4u+36 \\ 2r+4 & rt-17 & -19r+3u-11 \\ s+1 & t-4s+5 & su-8 \end{pmatrix}.$$

We know that $\mathbf{B} = \mathbf{A}^{-1} \iff \mathbf{AB} = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Inspection of the elements in the first row and the first column of $\mathbf{AB}$ shows that, if $\mathbf{AB} = \mathbf{I}$, then

$$r = -2, \quad s = -1, \quad t = -9, \quad u = -9\,.$$

It is easy to show that with these values of r, s, t, and u, the remaining elements of $\mathbf{AB}$ are also equal to the corresponding elements of $\mathbf{I}$.

Problem 4

(a) With $u = 1 + te^t$ we get $du = (e^t + te^t)\,dt = e^t(1+t)\,dt$ and

$$\begin{aligned}\int \frac{t+1}{t(1+te^t)}\,dt &= \int \frac{t+1}{t(1+te^t)}\,\frac{1}{e^t(1+t)}\,du = \int \frac{1}{te^t(1+te^t)}\,du \\ &= \int \frac{1}{(u-1)u}\,du = \int \Big(\frac{1}{u-1} - \frac{1}{u}\Big)du \\ &= \ln|u-1| - \ln|u| + C = \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{te^t}{1+te^t}\right| + C\,.\end{aligned}$$

(*Comment:* It can be shown that $1 + te^t$ is positive for all t, but that is not important in this problem.)

(b) The equation is separable. It has one constant solution, namely $x \equiv 0$. The nonconstant solutions are determined by the standard procedure of separation and integration:

$$\begin{aligned}\frac{\dot{x}}{x^2} &= \frac{1+t}{t(1+te^t)} \\ \int \frac{1}{x^2}\,dx &= \int \frac{1+t}{t(1+te^t)}\,dt \\ -\frac{1}{x} &= \ln\left|\frac{te^t}{1+te^t}\right| + C \qquad \text{(from part (a))} \\ x &= -\frac{1}{\ln\left|\dfrac{te^t}{1+te^t}\right| + C}\end{aligned}$$

(c) One way to solve this problem is to determine the constant C in the solution above such that the corresponding solution curve passes through $(1,1)$, and then differentiate x to find the slope of the tangent. (The desired value of C turns out to be $C = \ln(1+e) - 2$.) But this involves some messy computation with possibilities for mistakes.

A simpler solution is to determine the slope directly from the differential equation. At $(1,1)$ we get $(1+e)\dot{x} = 2$, so the slope of the tangent is $2/(1+e)$. The equation for the tangent is then

$$x - 1 = \frac{2}{1+e}(t-1) \iff x = \frac{2}{1+e}\,t + \frac{e-1}{1+e}\,.$$

(In this equation (t,x) are the coordinates of an arbitrary point on the tangent.)