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Answers to the examination problems in ECON3120/4120 Mathematics 2, 8 December 2008

Problem 1

(a) Cofactor expansion gives

$$|\mathbf{A}_t| = 0 - 0 + \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & t \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & t \end{vmatrix} = t - 2\,.$$

(We expand the original matrix along the first row, and the two determinants of order 3 are expanded along the second and third rows, respectively.)

The matrix $\mathbf{A}_t$ has an inverse if and only if $|\mathbf{A}| \neq 0$, i.e. if and only if $t \neq 2$.

(b) Direct calculation yields

$$\mathbf{A}_t + \mathbf{A}_s = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & t & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & s & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & 0 & t+s & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & (t+s)/2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = 2\mathbf{A}_{(t+s)/2}$$

Since this is a 4×4 matrix, the result in part (a) implies that

$$|\mathbf{A}_t + \mathbf{A}_s| = 2^4|\mathbf{A}_{(t+s)/2}| = 16\big((t+s)/2 - 2\big) = 8t + 8s - 32\,.$$

(c) Since $|\mathbf{A}_t| \neq 0$ for all $t \neq 2$, Cramer's rule tells us that the equation system has a (unique) solution for such t. With $t = 2$, Gaussian elimination yields

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ -1 \\ \ \end{matrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} \ \\ -1 \\ \ \\ \leftarrow \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix} \begin{matrix} \ \\ \ \\ -2 \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The last row in the final matrix corresponds to the impossible equation $0 = -1$, so in this case the equation system has no solution.

M2xh08fasit 11.12.2008 890

Of course, we could start solving the system in a more or less systematic way without using formal Gaussian elimination, but the result would be the same: we would get an impossible equation, and so the given system has no solution if $t = 2$.

Problem 2

Taking differentials, we get the equations

$$e^x y\,dx + e^x\,dy + du - dv = 0$$
$$dx - e^{u^2+v}(2\,du + dv) + dy = 0$$

Inserting the values of the variables at the point P_0, we get

$$\begin{aligned} dy + du - dv &= 0 \\ dx - e(2\,du + dv) + dy &= 0 \end{aligned} \quad\Longleftrightarrow\quad \begin{aligned} du - dv &= -dy \\ 2\,du + dv &= \tfrac{1}{e}\,(dx + dy) \end{aligned}$$

Solving these equations for du and dv yields

$$du = \frac{1}{3e}\,dx + \frac{1-e}{3e}\,dy\,, \quad dv = \frac{1}{3e}\,dx + \frac{2e+1}{3e}\,dy\,.$$

Hence,

$$u'_x = \frac{1}{3e}\,, \quad u'_y = \frac{1-e}{3e}\,, \quad v'_x = \frac{1}{3e}\,, \quad v'_y = \frac{2e+1}{3e}\,.$$

(Instead of taking differentials we could have used implicit differentiation with respect to each of x and y in the "usual" way, but that would lead to a little more work.)

(The problem only asks for the values of the partial derivatives of u and v at the particular point P_0, but it is likely that some students look for the values at a *general* point that solves the given equation system. The values of the partial derivatives at such a point are

$$u'_x = \frac{e^{-u^2-v} - e^x y}{3}\,, \qquad u'_y = \frac{e^{-u^2-v} - e^x}{3}\,,$$
$$v'_x = \frac{e^{-u^2-v} + 2e^x y}{3}\,, \qquad v'_y = \frac{e^{-u^2-v} + 2e^x}{3}\,.$$

But there is hardly any reason to give extra credit for calculating these expressions.)

M2xh08fasit 11.12.2008 890

Problem 3

The stationary points are where

$$f_1'(x,y) = \frac{10}{x+2y} + 1 - 3x + 6y = 0 \tag{1}$$

and

$$f_2'(x,y) = \frac{20}{x+2y} - 22 + 6x = 0\,. \tag{2}$$

Equation (1) implies $10/(x+2y) = 3x - 6y - 1$, and if we insert this in equation (2) we get

$$2(3x-6y-1) - 22 + 6x = 0 \iff 12x - 12y = 24 \iff x = y + 2\,.$$

Hence (equation (1) again),

$$\frac{10}{3y+2} + 3y - 5 = 0 \iff 10 + (3y+2)(3y-5) = 0 \iff 9y^2 - 9y = 0\,,$$

which has the solutions $y_1 = 0$ and $y_2 = 1$. Thus, the stationary points are $(x_1, y_1) = (2, 0)$ and $(x_2, y_2) = (3, 1)$.

To determine the nature of the stationary points we shall use the second-derivative test. The various second derivatives of f are

$$f_{11}''(x,y) = -\frac{10}{(x+2y)^2} - 3\,, \quad f_{12}''(x,y) = -\frac{20}{(x+2y)^2} + 6\,,$$
$$f_{22}''(x,y) = -\frac{40}{(x+2y)^2}\,.$$

With $A = f_{11}''(x,y)$, $B = f_{12}''(x,y)$, and $C = f_{22}''(x,y)$, the test gives

Point	A	B	C	$AC - B^2$	Result
$(2,0)$	$-\frac{11}{2}$	1	-10	54	Local max. point
$(3,1)$	$-\frac{17}{5}$	$\frac{26}{5}$	$-\frac{8}{5}$	$-\frac{108}{5}$	Saddle point

Problem 4

(a) We use formula (5) on page 334 in EMEA (page 13 in MA II) with $a = -1$ and $b(t) = e^t - t$, and get

$$x = Ce^t + e^t \int (1 - te^{-t})\, dt\,. \tag{$*$}$$

To evaluate the integral we use integration by parts on the second term:

$$\int (1 - te^{-t})\, dt = t - \int te^{-t}\, dt = t + te^{-t} - \int 1\, e^{-t}\, dt = t + te^{-t} + e^{-t} \quad (+\text{ const.})$$

(The constant of integration is already taken care of by C in $(*)$.) Inserting this integral into $(*)$ we get

$$x = Ce^t + te^t + t + 1\,.$$

(b) The solution will pass through $(t_0, x_0) = (1, 2)$ if and only if C is such that

$$2 = Ce + e + 1 + 1\,, \quad \text{i.e.} \quad C = -1\,.$$

Thus the desired solution is $x = (t-1)e^t + t + 1$, and K is the graph of this solution in the tx-plane. The derivative of x is $\dot{x} = te^t + 1$, so the slope of the tangent to K at $(t_0, x_0) = (1, 2)$ is $a = e + 1$. Hence the equation of the tangent is

$$x - 2 = (e+1)(t-1) \quad \text{or, equivalently,} \quad x = (e+1)t - e + 1.$$

A point (t, x) belongs to both K and this tangent if and only if

$$x = (t-1)e^t + t + 1 \quad \text{and} \quad x = (e+1)t - e + 1\,.$$

These equations imply

$$\begin{aligned}(t-1)e^t + t + 1 &= (e+1)t - e + 1 = (t-1)e + t + 1 \\ &\iff (t-1)e^t = (t-1)e \iff (t-1)(e^t - e) = 0.\end{aligned}$$

The last equation is satisfied for $t = 1$ but not for any other value of t. (For if $t \neq 1$, then $e^t \neq e$ too, and then $(t-1)(e^t - e) \neq 0$.) Thus $(t, x) = (1, 2)$ is the only point that lies on both K and the tangent we found above.

M2xh08fasit 11.12.2008 890