Mathematics 2 – sample problem set for December 7th 2011

- For this problem set, take the Autumn 2010 set (which is part of this PDF file) and *remove problem 3.* Instead
  - Do one of the two below given problems. The first is taken from Spring 2008 – if you have seen it already, I suggest that for a more realistic exam-alike setting, you do the second alternative.
- Both these problems (in addition to the entire Autumn 2010 set) will be reviewed in the December 7th lecture.

Problem 3, alternative (I) taken from Spring 2008 problem 1 Let for each d the matrix $\mathbf{A}_d$ be given by

$$\mathbf{A}_d = \begin{pmatrix} d^2 & 1 & 2 & 3 & 4 \\ 0 & -1 & 5 & 6 & 7 \\ 0 & 0 & d^2+1 & 8 & 9 \\ 0 & 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 0 & d^2 \end{pmatrix}$$

(a) Calculate $|\mathbf{A}_d|$ and show that the nth power $\mathbf{A}_d^n$ has an inverse if and only if $d \neq 0$.

(b) Consider the equation system

$$\mathbf{A}_d^{2008}\,\mathbf{x} = \begin{pmatrix} d \\ d \\ d \\ d \\ d \end{pmatrix}$$

where $\mathbf{x}$ is a vector $(x_1, \ldots, x_5)'$ of five unknowns and $\mathbf{A}_d^{2008}$ is the 2008th power of $\mathbf{A}_d$.

(i) For what values of d will the equation system have a *unique* solution?

(ii) Show that there always is a solution.

You may use the result given in part (a) no matter whether you managed to show it or not.

Problem 3, alternative II Define for $x \geq 0$, $y \geq 0$ the function $F(x, y) = xy + y^{1/4}$.

(a) The equation $F(x, y) = C$ defines y as a function of x. Find an expression for the elasticity of y with respect to x.

(b) Find an expression for the elasticity of substitution.

University of Oslo / Department of Economics **English**

ECON3120/ECON4120 Mathematics 2

Monday December 13 2010, 09:00–12:00

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

You are required to state reasons for all your answers. Throughout the problem set, you are permitted to use without proof information given in a previous part (e.g. in problem 3 (b), you can use the information given in part (a), regardless of whether you managed to solve it or not).

Problem 1 Let r be a constant, and consider the differential equation

$$\dot{x}(t) + 2x(t) = te^{-rt}.$$

(a) Find the general solution.

(b) For each of the values $r_1 = -e$ and $r_2 = e$ for the constant r, find the particular solution which passes through the origin, and check whether it tends to a limit as $t \to +\infty$.

Problem 2 *(Part (b) will have more weight than part (a) when grading.)*
Assume that the equation system

$$y + 2x - s + te^{-t} + 1 = 0$$
$$x + y + e^{y+s} - \ln(1 + t^2) = 0$$

defines x and y as continuously differentiable functions of (s, t).

(a) Differentiate the equation system (i.e. calculate differentials).

(b) Find expressions for x'_s and x'_t and show that $x''_{st} = 0$.

Problem 3 *(Part (b) will have less weight than part (a) when grading.)*
Throughout this problem, k will be a fixed positive integer, $\mathbf{A}$ will be an $n \times n$ matrix, and $\mathbf{I}$ will be the identity matrix of the same order. Define

$$\mathbf{B} = \mathbf{I} - \mathbf{A} \qquad \text{and} \qquad \mathbf{C} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k$$

(a) Calculate the product $\mathbf{BC}$ and show that $\mathbf{C}$ is the inverse of $\mathbf{B}$ *if and only if* $\mathbf{A}^{k+1} = \mathbf{0}$.

(b) If $\mathbf{C}$ is the inverse of $\mathbf{B}$, what do we then know about the number of solutions of the equation system $\mathbf{Ax} = \mathbf{0}$? (Here, $\mathbf{x}$ is the unknown.)

Problem 4 Let $f(x, y) = xy \ln(1 + xy)$, and consider the problems

$$\max f(x,y) \quad \text{subject to} \quad (x-1)^2 + (y-1)^2 \leq 1 \tag{K}$$

$$\max f(x,y) \quad \text{subject to} \quad (x-1)^2 + (y-1)^2 = 1 \tag{L}$$

(Notice «$\leq$» in (K) and «$=$» in (L).)

(a) Show that each of these problems has a solution, and state the Kuhn–Tucker conditions associated to problem (K) and the Lagrange conditions associated to problem (L).

(b) For each of the three points $(0, 1)$, $(1 - \frac{1}{2}\sqrt{2}, 1 - \frac{1}{2}\sqrt{2})$ and $(1 + \frac{1}{2}\sqrt{2}, 1 + \frac{1}{2}\sqrt{2})$, show that it satisfies the Lagrange conditions associated to problem (L), and check whether it satisfies the Kuhn–Tucker condition associated to problem (K).

(c) It can be shown – but you are not supposed to do so – that problem (L) has solution for the point $(x, y) = (q, q)$, where $q = 1 + \frac{1}{2}\sqrt{2}$. Find an approximation for the maximum value of f subject to the constraint $(x - 1)^2 + (y - 1)^2 = 0.98$. You can express the answer in terms of q and/or $f(q, q)$ without calculating these quantities.