ECON3120/ECON4120 Mathematics 2

Monday December 13 2010, 09:00–12:00

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

You are required to state reasons for all your answers. Throughout the problem set, you are permitted to use without proof information given in a previous part (e.g. in problem 3 (b), you can use the information given in part (a), regardless of whether you managed to solve it or not).

Problem 1 Let r be a constant, and consider the differential equation

$$\dot{x}(t) + 2x(t) = te^{-rt}.$$

(a) Find the general solution.

(b) For each of the values $r_1 = -e$ and $r_2 = e$ for the constant r, find the particular solution which passes through the origin, and check whether it tends to a limit as $t \to +\infty$.

Problem 2 *(Part (b) will have more weight than part (a) when grading.)*
Assume that the equation system

$$\begin{aligned} y + 2x - s + te^{-t} + 1 &= 0 \\ x + y + e^{y+s} - \ln(1+t^2) &= 0 \end{aligned}$$

defines x and y as continuously differentiable functions of (s, t).

(a) Differentiate the equation system (i.e. calculate differentials).

(b) Find expressions for x'_s and x'_t and show that $x''_{st} = 0$.

Problem 3 *(Part (b) will have less weight than part (a) when grading.)*
Throughout this problem, k will be a fixed positive integer, $\mathbf{A}$ will be an $n \times n$ matrix, and $\mathbf{I}$ will be the identity matrix of the same order. Define

$$\mathbf{B} = \mathbf{I} - \mathbf{A} \qquad \text{and} \qquad \mathbf{C} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k$$

(a) Calculate the product $\mathbf{BC}$ and show that $\mathbf{C}$ is the inverse of $\mathbf{B}$ *if and only if* $\mathbf{A}^{k+1} = \mathbf{0}$.

(b) If $\mathbf{C}$ is the inverse of $\mathbf{B}$, what do we then know about the number of solutions of the equation system $\mathbf{Ax} = \mathbf{0}$? (Here, $\mathbf{x}$ is the unknown.)

Problem 4 Let $f(x, y) = xy \ln(1 + xy)$, and consider the problems

$$\max f(x,y) \quad \text{subject to} \quad (x-1)^2 + (y-1)^2 \leq 1 \tag{K}$$

$$\max f(x,y) \quad \text{subject to} \quad (x-1)^2 + (y-1)^2 = 1 \tag{L}$$

(Notice «$\leq$» in (K) and «$=$» in (L).)

(a) Show that each of these problems has a solution, and state the Kuhn–Tucker conditions associated to problem (K) and the Lagrange conditions associated to problem (L).

(b) For each of the three points $(0, 1)$, $(1 - \frac{1}{2}\sqrt{2}, 1 - \frac{1}{2}\sqrt{2})$ and $(1 + \frac{1}{2}\sqrt{2}, 1 + \frac{1}{2}\sqrt{2})$, show that it satisfies the Lagrange conditions associated to problem (L), and check whether it satisfies the Kuhn–Tucker condition associated to problem (K).

(c) It can be shown – but you are not supposed to do so – that problem (L) has solution for the point $(x, y) = (q, q)$, where $q = 1 + \frac{1}{2}\sqrt{2}$. Find an approximation for the maximum value of f subject to the constraint $(x - 1)^2 + (y - 1)^2 = 0.98$. You can express the answer in terms of q and/or $f(q, q)$ without calculating these quantities.