University of Oslo / Department of Economics/NCF

ECON3120/ECON4120 Mathematics 2 – the 20101213 exam solved

Weighting: Suggested, and carried out: 150% for part 2(b) and 50% for part 3(b).

Problem 1 Let r be a constant, and consider the differential equation

$$\dot{x}(t) + 2x(t) = te^{-rt}.$$

(a) Find the general solution.

(b) For each of the values $r_1 = -e$ and $r_2 = e$ for the constant r, find the particular solution which passes through the origin, and check whether it tends to a limit as $t \to +\infty$.

Annotated solution:

(a) By integrating factor (one could alternatively use the formula), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}(xe^{2t}) = (\dot{x} + 2x)e^{2t} = te^{-rt} \cdot e^{2t} = te^{(2-r)t}$$

so that

$$xe^{2t} = \int te^{(2-r)t}\,\mathrm{d}t = \begin{cases} \frac{1}{2-r}te^{(2-r)t} - \int \frac{1}{2-r}e^{(2-r)t}\,\mathrm{d}t = \frac{1}{2-r}\big(t - \frac{1}{2-r}\big)e^{(2-r)t} + C, & r \neq 2 \\ \frac{1}{2}t^2 + C, \qquad r = 2. \end{cases}$$

leading to the general solution

$$x(t) = \begin{cases} Ce^{-2t} + \frac{1}{2-r}\big(t - \frac{1}{2-r}\big)e^{-rt}, & r \neq 2 \\ Ce^{-2t} + \frac{1}{2}t^2e^{-2t}, & r = 2. \end{cases}$$

(b) Both these values for r are distinct from 2, so with $x = t = 0$, we have $0 = C - (2-r)^{-2}$. The respective particular solutions become $\frac{1}{(2-r)^2} \cdot \big[e^{-2t} + \big(t(2-r) - 1\big)e^{-rt}\big]$, i.e.

$$\frac{1}{(2+e)^2} \cdot \big[e^{-2t} + \big(t(2+e) - 1\big)e^{et}\big], \text{resp.}$$
$$\frac{1}{(2-e)^2} \cdot \big[e^{-2t} + \big(t(2-e) - 1\big)e^{-et}\big]$$

Their respective limits are $+\infty$ and – because exponential decay kills polynomial growth – 0.

Remarks: It was intentional to have the oddball case $r = 2$ included in (a) (having stressed division-by-zero during the course), and excluded from (b).

Problem 2 *(Part (b) will have more weight than part (a) when grading.)*
Assume that the equation system

$$y + 2x - s + te^{-t} + 1 = 0$$
$$x + y + e^{y+s} - \ln(1 + t^2) = 0$$

defines x and y as continuously differentiable functions of (s, t).

(a) Differentiate the equation system (i.e. calculate differentials).

(b) Find expressions for x'_s and x'_t and show that $x''_{st} = 0$.

Annotated solution:

(a) The differentiated system becomes

$$\mathrm{d}y + 2\,\mathrm{d}x - \mathrm{d}s + (e^{-t} - te^{-t})\,\mathrm{d}t = 0$$
$$\mathrm{d}x + \mathrm{d}y + e^{y+s}(\mathrm{d}y + \mathrm{d}s) - \frac{2t}{1+t^2}\,\mathrm{d}t = 0$$

(One might gather the $\mathrm{d}y$ terms.)

(b) The probably shortest way is to eliminate $\mathrm{d}y$ from the first equation. Subtracting $(1 + e^{y+s})$ of this from the second, yields

$$0 = [1 - 2(1 + e^{y+s})]\,\mathrm{d}x + [e^{y+s} + (1 + e^{y+s})]\,\mathrm{d}s + [(1 + e^{y+s})(t-1)e^{-t} - \frac{2t}{1+t^2}]\,\mathrm{d}t$$
$$= -[1 + 2e^{y+s}][\mathrm{d}x - \mathrm{d}s] - [(1 + e^{y+s})(1-t)e^{-t} + \frac{2t}{1+t^2}]\,\mathrm{d}t$$

so that

$$x'_s = 1 \qquad \text{implying } x''_{st} = 0$$
$$x'_t = -\frac{(1 + e^{y+s})(1-t)e^{-t} + \frac{2t}{1+t^2}}{1 + 2e^{y+s}}$$

Remarks: In the lectures, the method of writing the differentiated system as

$$\mathbf{M}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} = \begin{pmatrix}\mathrm{d}s + (t-1)e^{-t}\,\mathrm{d}t\\ -e^{y+s}\,\mathrm{d}s + \frac{2t}{1+t^2}\,\mathrm{d}t\end{pmatrix}$$

was taught as well. If done by inversion, the calculations do become somewhat lengthy (Cramér's rule may simplify).

Problem 3 *(Part (b) will have less weight than part (a) when grading.)*
Throughout this problem, k will be a fixed positive integer, $\mathbf{A}$ will be an $n \times n$ matrix, and $\mathbf{I}$ will be the identity matrix of the same order. Define

$$\mathbf{B} = \mathbf{I} - \mathbf{A} \qquad \text{and} \qquad \mathbf{C} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k$$

(a) Calculate the product $\mathbf{BC}$ and show that $\mathbf{C}$ is the inverse of $\mathbf{B}$ *if and only if* $\mathbf{A}^{k+1} = \mathbf{0}$.

(b) If $\mathbf{C}$ is the inverse of $\mathbf{B}$, what do we then know about the number of solutions of the equation system $\mathbf{Ax} = \mathbf{0}$? (Here, $\mathbf{x}$ is the unknown.)

Annotated solution:

(a) The product $\mathbf{BC}$ becomes

$$\begin{aligned} \mathbf{C} - \mathbf{AC} &= \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{k-1} + \mathbf{A}^k \\ &\quad - (\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^k + \mathbf{A}^{k+1}) \\ &= \mathbf{I} - \mathbf{A}^{k+1} \end{aligned}$$

Since $\mathbf{B}$ and $\mathbf{C}$ are square, they are inverses of each other if and only if the product is $\mathbf{I}$, i.e. if and only if $\mathbf{A}^{k+1} = \mathbf{0}$.

(b) If $\mathbf{C}$ is the inverse of $\mathbf{B}$, then from (a) we know that $\mathbf{A}^{k+1} = \mathbf{0}$, which implies that $0 = |\mathbf{A}^{k+1}| = |\mathbf{A}|^{k+1}$ and thus $|\mathbf{A}| = 0$. Therefore the equation has infinitely many solutions (it has at least one solution since the right hand side is null).

Problem 4 Let $f(x, y) = xy \ln(1 + xy)$, and consider the problems

$$\max f(x,y) \quad \text{subject to} \quad (x-1)^2 + (y-1)^2 \le 1 \qquad \text{(K)}$$
$$\max f(x,y) \quad \text{subject to} \quad (x-1)^2 + (y-1)^2 = 1 \qquad \text{(L)}$$

(Notice «$\le$» in (K) and «$=$» in (L).)

(a) Show that each of these problems has a solution, and state the Kuhn–Tucker conditions associated to problem (K) and the Lagrange conditions associated to problem (L).

(b) For each of the three points $(0, 1)$, $(1 - \frac{1}{2}\sqrt{2}, 1 - \frac{1}{2}\sqrt{2})$ and $(1 + \frac{1}{2}\sqrt{2}, 1 + \frac{1}{2}\sqrt{2})$, show that it satisfies the Lagrange conditions associated to problem (L), and check whether it satisfies the Kuhn–Tucker condition associated to problem (K).

(c) It can be shown – but you are not supposed to do so – that problem (L) has solution for the point $(x, y) = (q, q)$, where $q = 1 + \frac{1}{2}\sqrt{2}$. Find an approximation for the maximum value of f subject to the constraint $(x - 1)^2 + (y - 1)^2 = 0.98$. You can express the answer in terms of q and/or $f(q, q)$ without calculating these quantities.

Annotated solution:

(a) Each constraint forms a closed bounded set, so the extreme value theorem implies solution.
Define the Lagrangian

$$L(x,y) = f(x,y) - \lambda((x-1)^2 + (y-1)^2 - 1).$$

The Lagrange conditions are then

$$\big[\ln(1+xy) + \frac{xy}{1+xy}\big]y - 2\lambda(x-1) = 0 \tag{1}$$

$$\big[\ln(1+xy) + \frac{xy}{1+xy}\big]x - 2\lambda(y-1) = 0 \tag{2}$$

$$(x-1)^2 + (y-1)^2 = 1. \tag{3}$$

The Kuhn–Tucker conditions are (1), (2) and

$$\lambda \geq 0 \quad (\text{with } \lambda = 0 \text{ if } (x-1)^2+(y-1)^2 < 1). \tag{3'}$$

(b)

- Point $(0,1)$ in the Lagrange conditions: (1) yields $\lambda = 0$, (2) yields $0 = 0$ and (3) yields $1^2 + 0^2 = 1$. OK.
  - Kuhn–Tucker conditions: (3') holds since $\lambda = 0$.
- Point $(1 - \frac{1}{2}\sqrt{2}, 1 - \frac{1}{2}\sqrt{2})$ in the Lagrange conditions:
  The LHS of (3) becomes $2(\frac{1}{2}\sqrt{2})^2$ which equals 1. (2) is the same as (1) since $x = y$, and since $(x-1)$ is nonzero, then (1) and (2) are satisfied by putting

  $$\lambda = (\ln(1+xy) + \frac{xy}{1+xy})\frac{y}{2(x-1)}.$$

  - Kuhn–Tucker conditions: It suffices to check the sign of λ. We have

  $$\lambda = \underbrace{(\ln(1+xy) + \frac{xy}{1+xy})}_{>0 \text{ since } xy>0} \frac{y}{2(x-1)}.$$

  Since $y > 0$, the sign is the same as of $x - 1$, which is negative. The point fails the Kuhn–Tucker conditions.
- Point $(1 + \frac{1}{2}\sqrt{2}, 1 + \frac{1}{2}\sqrt{2})$ in the Lagrange conditions: This is the same argument as the previous point.
  The LHS of (3) becomes $2(\frac{1}{2}\sqrt{2})^2$ which equals 1. (2) is the same as (1) since $x = y$, and since $(x-1)$ is nonzero, then (1) and (2) are satisfied by putting

  $$\lambda = (\ln(1+xy) + \frac{xy}{1+xy})\frac{y}{2(x-1)}. \tag{*}$$

– Kuhn–Tucker conditions: Again it suffices to check the sign of λ. We have

$$\lambda = \underbrace{(\ln(1 + xy) + \frac{xy}{1 + xy})}_{>0 \text{ since } xy>0} \frac{y}{2(x - 1)}.$$

Since $y > 0$, the sign is the same as of $x - 1$, which is positive. The point satisfies the Kuhn–Tucker conditions.

(c) The new value function is approximately equal to

$$f(q, q) + \lambda \cdot (-0.02) = f(q, q) - 0.02 \cdot (\ln(1 + q^2) + \frac{q^2}{1 + q^2})\frac{q}{2(q - 1)}$$

by inserting from (*).

Remarks: Some candidates might spot that maximizing f is equivalent to maximizing xy. However, they are expected to treat the problem as is, not rewrite it before e.g. stating conditions.

(a) The students have been explicitely allowed to deviate from the usual terminology by including the constraint in the Kuhn–Tucker conditions. I.e., (3') would then be augmented with e.g. «and $(x - 1)^2 + (y - 1)^2 \leq 1$».

(b) Part of the intention of asking for point $(0, 1)$, is to test whether it is understood that active constraint does not imply nonzero multiplier.
Part 4(b) might be just as lengthy as part 2(b), where the latter is to be weighted heavier. It is *suggested* to keep 4(b) as same weight as the other problems (despite the workload), in order not to penalize too heavily the calculations errors which are likely to occur.

(c) The numerical values are approximately $f(q, q) \approx 3.98$ and $\lambda \approx 2.55$, yielding a new value of approximately 3.93.