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Answers to the examination problems in ECON3120/4120 Mathematics 2, 2 June 2010

Problem 1

(a) The maximum point must satisfy the Lagrange conditions, and with the Lagrangian $\mathcal{L}(x,y) = \ln(1+x) + 3\ln(1+y) - \lambda(ax+y-m)$ the first-order conditions become

$$\mathcal{L}_1'(x,y) = \frac{1}{1+x} - \lambda a = 0\,, \tag{1}$$

$$\mathcal{L}_2'(x,y) = \frac{3}{1+y} - \lambda = 0\,. \tag{2}$$

The constraint is

$$ax + y = m\,. \tag{3}$$

Equation (2) implies $\lambda = \dfrac{3}{1+y}$, and then (1) yields

$$\frac{1}{1+x} = \lambda a = \frac{3a}{1+y}\,.$$

Hence,

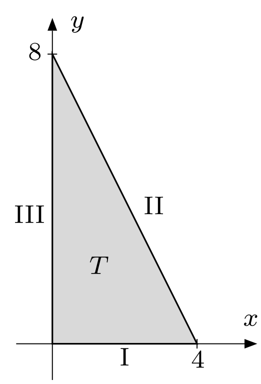
$$1 + y = 3a(1+x) = 3a + 3ax \iff -3ax + y = 3a - 1\,.$$

Together with (3) this yields

$$x = \frac{m - 3a + 1}{4a}\,, \quad y = \frac{3a + 3m - 1}{4}\,.$$

This is the only point that satisfies the Lagrange conditions, and since we know that there exists a maximum point, this point must be it.

(b) The set T is the triangular region with corners at $(0,0)$, $(4,0)$, and $(0,8)$. All points on the sides I, II, III of the triangle belong to T, so T is a closed set. It is also bounded and f is continuous, so the extreme value theorem guarantees that f will attain both a maximum and a minimum over T. Because f has no stationary points, the extreme points must be on the boundary of T. It is also clear that f is strictly increasing with respect to each of the variables, so $(0,0)$ is the unique minimum point, and the maximum point must be somewhere on II.



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The points on II all belong to the straight line $2x + y = 8$, and it follows from part (a) that the maximum of $f(x, y)$ along that line (that is, on the part where $x > -1$ and $y > -1$ so that f is defined) is attained at the point $(x^*, y^*) = (3/8, 29/4)$. This point obviously belongs to the line segment II, and it is therefore the maximum point for f over T. The extreme *values* of f are then

$$\begin{aligned} f_{\text{maks}} &= f(x^*, y^*) = f\Big(\frac{3}{8}, \frac{29}{4}\Big) = \ln\Big(\frac{11}{8}\Big) + 3\ln\Big(\frac{33}{4}\Big) \approx 6.6490933\,, \\ f_{\min} &= f(0, 0) = 0\,. \end{aligned}$$

(The expression for f_{maks} can be simplified to $4\ln 11 + 3\ln 3 - 9\ln 2$, but that is not necessary.)

Problem 2

(a) The result follows immediately from $\dfrac{d}{dx}(\ln f(x)) = \dfrac{f'(x)}{f(x)}$.

(b) The standard procedure yields

$$\begin{aligned} \frac{e^x}{1+e^x}\dot{x} &= \frac{2t}{1+t^2} \\ \int \frac{e^x}{1+e^x}\,dx &= \int \frac{2t}{1+t^2}\,dt \\ \ln(1+e^x) &= \ln(1+t^2) + C_1 && \text{(by part (a))} \\ 1 + e^x &= C(1+t^2) && \text{(with } C = e^{C_1}\text{)} \end{aligned}$$

Solving for x yields $x = \ln(C(1+t^2) - 1)$. There are no constant solutions.

(c) We need to find the tangent to the solution curve at $(t_0, x_0) = (1, 0)$. The slope of the tangent can be found directly from the differential equation, since

$$\dot{x} = \frac{2t(1+e^x)}{(1+t^2)e^x}$$

With $t = 1$ and $x = 0$ this gives $\dot{x} = 2$. Thus the tangent is given by the equation

$$x - 0 = 2(t - 1) \iff x = 2t - 2\,,$$

and this equation is satisfied at $(t, x) = (2, 2)$.

Alternatively, we can first determine the solution curve through (t_0, x_0). Then we need to find the corresponding value of C:

$$x_0 = \ln(C(1+t_0^2) - 1) \iff 0 = \ln(2C - 1) \iff 2C - 1 = 1 \iff C = 1\,.$$

Thus the solution curve in question is $x = \ln(1 + t^2 - 1) = 2\ln t$, and $\dot{x} = 2/t$, etc.

Problem 3

(a) The derivative of S is given by

$$S'(t) = C(-ae^{-at})(e^{-at}+b)^{-2} + Ce^{-at}(-2)(e^{-at}+b)^{-3}(-ae^{-at})$$
$$= \frac{2aCe^{-2at}}{(e^{-at}+b)^3} - \frac{aCe^{-at}}{(e^{-at}+b)^2} = \cdots = \frac{aCe^{-at}(e^{-at}-b)}{(e^{-at}+b)^3}.$$

(b) We see from the answer in part (a) that

$$S'(t^*) = 0 \iff e^{-at^*} = b \iff -at^* = \ln b \iff t^* = -(\ln b)/a\,.$$

The sign of $S'(t)$ is the same as the sign of the factor $e^{-at}-b$. This factor is strictly decreasing with respect to t, so $S'(t) > 0$ for $t < t^*$ and $S'(t) < 0$ for $t > t^*$. Thus, S is strictly increasing in the interval $(-\infty, t^*]$ and strictly decreasing in $[t^*, \infty)$, and it follows that t^* is a global maximum point for S.

(c) Since a and b are positive, $t^* > 0 \iff \ln b < 0 \iff 0 < b < 1$.

(d) The substitution $u = e^{-at} + b$ yields $du = -ae^{-at}\,dt$ and

$$\int S(t)\,dt = \int C\frac{e^{-at}}{(e^{-at}+b)^2}\,dt = -\frac{C}{a}\int \frac{1}{u^2}\,du = \frac{C}{au} + K = \frac{C}{a(e^{-at}+b)} + K\,,$$

where K is the constant of integration. It follows that

$$\int_0^T S(t)\,dt = \Big|_0^T \frac{C}{a(e^{-at}+b)}\,dt = \frac{C}{a}\Big(\frac{1}{e^{-aT}+b} - \frac{1}{1+b}\Big) = \frac{C}{a}\,\frac{1-e^{-aT}}{(e^{-aT}+b)(b+1)}\,,$$

and

$$\int_0^\infty S(t)\,dt = \lim_{T\to\infty}\int_0^T S(t)\,dt = \frac{C}{ab(b+1)}\,,$$

because $e^{-aT} \to 0$ as $T \to \infty$.

Problem 4

(a) Cofactor expansion along the first row gives

$$|\mathbf{A}_t| = 0\cdot(\cdots) - 1\begin{vmatrix} 1 & -t \\ t-1 & 1 \end{vmatrix} + t\begin{vmatrix} 1 & 0 \\ t-1 & 1 \end{vmatrix} = -(1+t^2-t)+t = -t^2+2t-1\,.$$

(b) (i) With $t = 1$ the system becomes

$$\begin{aligned} y + z &= 0 \\ x \qquad - z &= 0 \\ y + z &= 0 \end{aligned} \iff \begin{aligned} y &= -z \\ x &= z \end{aligned}$$

and the solutions of the system are $(x, y, z) = (s, -s, s)$ for all real numbers s.

(ii) If $t = 2$, then $|\mathbf{A}_t| = -1 \neq 0$ and the system has only the trivial solution $(x, y, z) = (0, 0, 0)$.

(c) If we both premultiply and postmultiply by $\mathbf{B}^{-1}$ in the equation $\mathbf{BC} = \mathbf{CB}$, we get

$$\mathbf{B}^{-1}(\mathbf{BC})\mathbf{B}^{-1} = \mathbf{B}^{-1}(\mathbf{CB})\mathbf{B}^{-1} \iff \mathbf{ICB}^{-1} = \mathbf{B}^{-1}\mathbf{CI} \iff \mathbf{CB}^{-1} = \mathbf{B}^{-1}\mathbf{C}\,,$$

and the last equation shows that $\mathbf{B}^{-1}$ and $\mathbf{C}$ commute with each other.

Problem 5

By the rule $\ln a^p = p \ln a$, we have $\ln f(x) = \dfrac{1}{\ln(e^x - 1)} \ln x = \dfrac{\ln x}{\ln(e^x - 1)}$, and l'Hôpital's rule gives

$$\lim_{x\to 0^+} \ln f(x) = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to 0^+} \frac{\dfrac{1}{x}}{\dfrac{e^x}{e^x - 1}} = \lim_{x\to 0^+} \frac{e^x - 1}{xe^x} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 0^+} \frac{e^x}{e^x + xe^x} = 1\,.$$

It follows that $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} e^{\ln f(x)} = e^1 = e$.