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Answers to the examination problems in ECON3120/4120 Mathematics 2, 31 May 2011

Problem 1

(a) Cofactor expansion along the bottom row yields

$$|\mathbf{A}_t| = 1\begin{vmatrix} 2 & 1 \\ t & 0 \end{vmatrix} - 0\cdot(\cdots) + t\begin{vmatrix} 4+t & 2 \\ 2 & t \end{vmatrix} = -t + t(4t + t^2 - 4) = t^3 + 4t^2 - 5t\,.$$

It follows that $|\mathbf{A}_0| = 0$. To determine all t for which $|\mathbf{A}_t| = 0$ we need to solve the cubic equation $t^3 + 4t^2 - 5t = 0$. We can write the equation as

$$t(t^2 + 4t - 5) = 0\,,$$

and it follows that the roots are $t = 0$ (which we already know) together with the roots of the quadratic equation $t^2 + 4t - 5 = 0$, namely $t = 1$ and $t = -5$.

Comment: We could of course have tackled $|\mathbf{A}_0|$ directly as $|\mathbf{A}_0| = \begin{vmatrix} 4 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}$, which is 0 because the last two rows (or the last two columns) are proportional. But we needed the general determinant $|\mathbf{A}_t|$ for the last question anyway.

(b) We use Gaussian elimination:

$$\begin{pmatrix} 5 & 2 & 1 & a \\ 2 & 1 & 0 & b \\ 1 & 0 & 1 & c \end{pmatrix} \begin{matrix} \leftarrow \\ \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 0 & 1 & c \\ 2 & 1 & 0 & b \\ 5 & 2 & 1 & a \end{pmatrix} \begin{matrix} -2 \quad -5 \\ \leftarrow \\ \leftarrow \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & c \\ 0 & 1 & -2 & b-2c \\ 0 & 2 & -4 & a-5c \end{pmatrix} \begin{matrix} \\ -2 \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 0 & 1 & c \\ 0 & 1 & -2 & b-2c \\ 0 & 0 & 0 & a-2b-c \end{pmatrix}$$

It is clear that the last matrix represents a system that has solutions if and only if $a - 2b - c = 0$.

Problem 2

(a) Differentiate the equation $x\varphi(x) + \varphi(x)^3 = 3$ with respect to x. That gives

$$\varphi(x) + x\varphi'(x) + 3\varphi(x)^2\varphi'(x) = 0\,, \tag{1}$$

and therefore

$$\varphi'(x) = -\frac{\varphi(x)}{x + 3\varphi(x)^2}\,. \tag{2}$$

M2xv11fasit 1.6.2011 566

(b) For the quadratic approximation we need the value of the second derivative $\varphi''(x)$ at $x = 2$. We can get $\varphi''(x)$ by taking the derivative of the fraction in (2), but it is simpler to differentiate the equation (1). We get

$$\varphi'(x) + \varphi'(x) + x\varphi''(x) + 6\varphi(x)\varphi'(x)^2 + 3\varphi(x)^2\varphi''(x) = 0\,,$$

$$\varphi''(x) = -\frac{2\varphi'(x) + 6\varphi(x)\varphi'(x)^2}{x + 3\varphi(x)^2}\,. \tag{3}$$

Since $\varphi(2) = 1$, it follows from (2) that $\varphi'(2) = -1/5$, and then (3) gives

$$\varphi''(2) = -\frac{-\frac{2}{5} + \frac{6}{25}}{2+3} = \frac{4}{125}\,.$$

The quadratic approximation to $\varphi(x)$ around $x_0 = 2$ is therefore

$$\varphi(2+h) \approx \varphi(2) + \varphi'(2)h + \frac{1}{2}\varphi''(2)h^2 = 1 - \frac{1}{5}h + \frac{2}{125}h^2\,.$$

Problem 3

(a) By formula (9.9.5) in EMEA (formula (1.4.5) in MA2), the general solution is

$$x = Ce^{-t/2} + e^{-t/2}\int e^{t/2}(2-t)\,dt. \tag{$\heartsuit$}$$

To evaluate the integral in ($\heartsuit$) we use integration by parts, integrating the first factor and then differentiating the other factor:

$$\begin{aligned}\int e^{t/2}(2-t)\,dt &= 2e^{t/2}(2-t) - \int 2e^{t/2}(-1)\,dt\\ &= 4e^{t/2} - 2te^{t/2} + 4e^{t/2} = 8e^{t/2} - 2te^{t/2}\end{aligned}$$

(the constant of integration is already taken care of in ($\heartsuit$).) Thus the general solution of the differential equation is

$$x = Ce^{-t/2} + e^{-t/2}(8e^{t/2} - 2te^{t/2}) = Ce^{-t/2} + 8 - 2t\,. \tag{$\diamondsuit$}$$

(b) At the point of tangency we must have $x = 0$ and $\dot{x} = 0$, so the differential equation $\dot{x} + \frac{1}{2}x = 2 - t$ gives $t = 2$. Thus the point of tangency is $(t, x) = (2, 0)$.

For a solution to pass through that point, the constant C in ($\diamondsuit$) must be such that $x(2) = 0$. In other words,

$$0 = Ce^{-1} + 8 - 4\,, \quad \text{i.e.} \quad C = -4e\,.$$

Thus the solution we are looking for is

$$x = -4e^{1-(t/2)} + 8 - 2t\,.$$

Problem 4

(a) We have

$$F'(t) = \frac{2-\ln t}{t^3} \begin{cases} > 0 & \text{if } 0 < t < e^2\,, \\ < 0 & \text{if } t > e^2\,. \end{cases}$$

Therefore F is strictly increasing on $(0, e^2]$ and strictly decreasing on $[e^2, \infty)$, so $F(t)$ attains its maximum for $t = e^2$.

To find an expression for $F(t)$ we first use integration by parts to find the indefinite integral

$$\begin{aligned} G(x) &= \int (2-\ln x)\frac{1}{x^3}\,dx = (2-\ln x)\Big(-\frac{1}{2x^2}\Big) - \int \Big(-\frac{1}{x}\Big)\Big(-\frac{1}{2x^2}\Big)\,dx \\ &= \frac{\ln x - 2}{2x^2} - \frac{1}{2}\int \frac{1}{x^3}\,dx = \frac{\ln x - 2}{2x^2} + \frac{1}{4x^2} + C = \frac{2\ln x - 3}{4x^2} + C\,. \end{aligned}$$

It follows that

$$F(t) = \Big|_1^t G(x) = \frac{2\ln t - 3}{4t^2} + \frac{3}{4}\,.$$

In particular,

$$F_{\max} = F(e^2) = \frac{2\cdot 2 - 3}{4e^4} + \frac{3}{4} = \frac{1}{4e^4} + \frac{3}{4}\,.$$

(b) By l'Hôpital's rule for "∞/∞" forms,

$$\lim_{t\to\infty} F(t) = \frac{3}{4} + \lim_{t\to\infty}\frac{2\ln t - 3}{4t^2} \overset{\text{l'Hôp}}{=} \frac{3}{4} + \lim_{t\to\infty}\frac{2/t}{8t} = \frac{3}{4} + \lim_{t\to\infty}\frac{1}{4t^2} = \frac{3}{4}\,.$$

Problem 5

(a) The first-order partial derivatives of f are

$$f_1'(x,y) = -y^3 - y^2 - 2x\,, \quad f_2'(x,y) = -3xy^2 - 2xy + 1\,.$$

At every point (x, y) in S we have $x > 0$ and $xy \geq 1$, so y is also positive. Therefore $f_1'(x,y) < 0$, and f has no stationary point in S. (It is also clear that $f_2'(x,y) < 0$ throughout S, because $f_2'(x,y) < -2xy + 1 \leq -2 + 1$.)

(b) Since f has no stationary point in S, the maximum point (or points) must lie on the boundary of S, i.e. on the curve $xy = 1$. Along that curve we have

$$f(x,y) = f(x, 1/x) = -\frac{x}{x^3} - \frac{x}{x^2} + \frac{1}{x} - x^2 = -x^{-2} - x^2\,.$$

The maximum points $(x, 1/x)$ for f over $xy = 1$ correspond to the maximum points of $g(x) = -x^{-2} - x^2$ over $(0, \infty)$. The derivative of g is $g'(x) = 2x^{-3} - 2x$ and the stationary points for g are the solutions of the equation

$$2x^{-3} - 2x = 0 \iff 2 = 2x^4 \iff x^4 = 1 \iff x^2 = 1\,.$$

The only positive solution is $x = 1$. Moreover, g is concave because $g''(x) = -6x^{-4} - 2 < 0$, so $x = 1$ is a maximum point (and the only one) for g over $(0, \infty)$. Hence, the unique maximum point for f over S is $(1, 1)$.