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# Answers to the examination problems in ECON3120/4120 Mathematics 2, 10 December 2009

## Problem 1

(a) Cofactor expansion along the first column gives

$$|\mathbf{A}| = (x+3)[(4-x)(-x)-12] = (x+3)(x^2-4x-12) = (x+3)(x+2)(x-6)\,,$$

so the matrix has an inverse provided $x \neq -3$, $x \neq -2$, and $x \neq 6$.

(b) The matrix $\mathbf{B}$ must be $2 \times 4$, and we see that $\mathbf{B} = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$.

(c) Transposing each side yields $\mathbf{C}^{-1} - 2\mathbf{I}_2 = -2\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, so

$$\mathbf{C}^{-1} = 2\mathbf{I}_2 - 2\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}.$$

By the formula for the inverse of a $2 \times 2$ matrix,

$$\mathbf{C} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

## Problem 2

(a) The Lagrangian is $\mathcal{L} = xy - \lambda((x+2a)(y+3a) - A)$, and the first-order conditions are:

$$\text{(i) } \mathcal{L}'_x = y - \lambda(y+3a) = 0 \qquad \text{(ii) } \mathcal{L}'_y = x - \lambda(x+2a) = 0$$

Eliminating $\lambda$, we get

$$\frac{y}{y+3a} = \frac{x}{x+2a}, \quad \text{that is,} \quad xy + 2ay = xy + 3ax, \quad \text{and so} \quad y = \tfrac{3}{2}x$$

Inserting $y = \frac{3}{2}x$ into the constraint yields the quadratic equation $x^2 + 4ax + 4a^2 - 2A/3 = 0$. The positive solution of this equation is $x = \sqrt{2A/3} - 2a$, and then $y = \frac{3}{2}(\sqrt{2A/3} - 2a) = \sqrt{3A/2} - 3a$.

(b) With $x^* = \sqrt{2A/3} - 2a$, $y^* = \frac{3}{2}(\sqrt{2A/3} - 2a)$ (with corresponding $\lambda = x^*/(x^*+2a) = 1 - a\sqrt{6/A}$)

$$f^*(a,A) = x^*y^* = \tfrac{3}{2}(x^*)^2 = \tfrac{3}{2}(\sqrt{2A/3} - 2a)^2 = 6a^2 - 2\sqrt{6}a\sqrt{A} + A$$

and thus $\partial f^*(a,A)/\partial A = -a\sqrt{6/A} + 1$, and $\partial f^*(a,A)/\partial a = 12a - 2\sqrt{6A}$.



M2xh09fasit01 27.11.2011 1208

(c) Writing the Lagrangian as $\mathcal{L}(x,y,A,a) = xy - \lambda((x+2a)(y+3a) - A)$, we get $\partial\mathcal{L}(x,y,A,a)/\partial A = \lambda$ (as expected) and $\partial\mathcal{L}(x,y,A,a)/\partial a = -\lambda(2(y+3a)+3(x+2a)) = -\lambda(12a+3x+2y)$. Evaluating these partials at $(x^*,y^*)$, we get $(\partial\mathcal{L}(x,y,A,a)/\partial A)_{(x^*,y^*)} = \lambda = 1 - a\sqrt{6/A}$ and $(\partial\mathcal{L}(x,y,A,a)/\partial a)_{(x^*,y^*)} = -\lambda(12a+3x^*+2y^*) = -(1-a\sqrt{6/A})(12a+6x^*) = -(1-a\sqrt{6/A})6\sqrt{2A/3} = -2\sqrt{6A}+12a$. So the envelope theorem confirms the results in (b).

## Problem 3

(a) Differentiation yields

$$\begin{aligned} du - 2v\,dv - 2\,dx - 2y\,dy &= 0 \\ e^{xu}(x\,du + u\,dx) + e^{yv}(y\,dv + v\,dy) &= 0 \end{aligned}$$

At $P$,

$$\begin{aligned} du - 2\,dv - 2\,dx - 2\,dy &= 0 \\ du + e\,dv + e\,dy &= 0 \end{aligned}$$

i.e.

$$\begin{aligned} du - 2\,dv &= 2\,dx + 2\,dy \\ du + e\,dv &= -e\,dy \end{aligned}$$

It follows that

$$du = \frac{2e}{e+2}\,dx\,, \qquad dv = \frac{-2}{e+2}\,dx - dy \tag{$*$}$$

(b) From $(*)$ we read off $\quad v'_x(1,1) = \dfrac{-2}{e+2}\,, \quad v'_y(1,1) = -1.$

(c)
$$\begin{aligned} v(0.99, 1.02) &\approx v(1,1) + v'_x(1,1)(-0.01) + v'_y(1,1)(0.02) \\ &= 1 + \frac{-2}{e+2}(-0.01) + (-1)0.02 \approx 1.016\,. \end{aligned}$$

## Problem 4

(a) (i) The derivative of the right-hand side equals the integrand on the left.

(ii) On the left we get $\displaystyle\int (x+kx)^{-1}\,dx = \int \frac{1}{(k+1)x}\,dx = \frac{\ln x}{k+1} + C_1\,.$

The limit of the integral in (i) as $r \to -1$ is

$$\lim_{r\to-1} \frac{\ln|k + x^{r+1}|}{r+1} + C$$

The denominator in the fraction tends to 0, and the numerator tends to $\ln|k+1|$, so the limit can exist only if $k=0$. Then

$$\lim_{r\to-1} \frac{\ln|k+x^{r+1}|}{r+1} + C = \lim_{r\to-1} \frac{\ln x^{r+1}}{r+1} + C = \lim_{r\to-1} \frac{(r+1)\ln x}{r+1} + C = \ln x + C\,,$$

which is the same as on the left.

(b) The equation is separable. It has one constant solution, given by $x = x^{2-e}$, i.e. $x = 1$. For $x \neq 1$ we get

$$\int \frac{1}{x - x^{2-e}}\, dx = \int \frac{2t}{e-1}\, dt.$$

Using the result in (a) (i) with $k = -1$ and $r = e - 2$ , we get

$$\frac{\ln|x^{e-1} - 1|}{e-1} = \frac{t^2}{e-1} + C\,,$$

i.e.

$$\ln|x^{e-1} - 1| = t^2 + C(e-1)\,.$$

This yields

$$x^{e-1} - 1 = C_2 e^{t^2}\,, \quad \text{and so} \quad x^{e-1} = 1 + C_2 e^{t^2}\,,$$

where $C_2 = \pm e^{C(e-1)}$. (Note that the constant solution would correspond to $C_2 = 0$.)

(c) The solution through $(e, e)$ is given by $C_2 = e^{-e^2}(e^{e-1} - 1)$, and the solution through $(1, 1)$ is the constant solution $x = 1$.