Department of Economics
A. Strøm

Answers to the examination problems in ECON 3120/4120, 24 November 2004

Problem 1

(a) $f'(x) = 2xe^{-bx} + (x^2 - a)(-be^{-bx}) = (-bx^2 + 2x + ab)e^{-bx}$,
$f''(x) = (-2bx + 2)e^{-bx} + (-bx^2 + 2x + ab)e^{-bx} = (b^2x^2 - 4bx + 2 - ab^2)e^{-bx}$.

(b) With $a = 5$ and $b = 1/2$, we get

$$f'(x) = (-\tfrac{1}{2}x^2 + 2x + \tfrac{5}{2})e^{-x/2} = -\tfrac{1}{2}(x^2 - 4x - 5)e^{-x/2},$$
$$f''(x) = (\tfrac{1}{4}x^2 - 2x + 2 - \tfrac{5}{4})e^{-x/2} = \tfrac{1}{4}(x^2 - 8x + 3)e^{-x/2}.$$

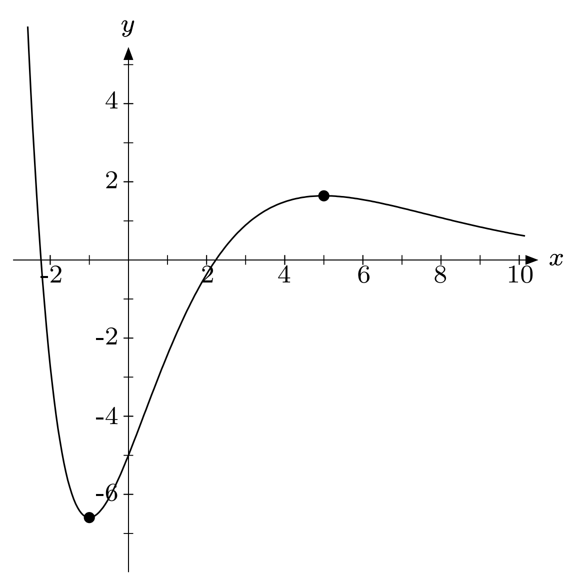
The stationary points of f are given by

$$f'(x) = 0 \iff x = -1 \text{ or } x = 5.$$

Further,

$$f''(-1) = 3\sqrt{e} > 0, \quad f''(5) = -3e^{-5/2} < 0.$$

Hence, $x = -1$ *is a local minimum point* for f and $x = 5$ *is a local maximum point.*



The graph of $f(x) = (x^2 - 5)e^{-x/2}$

M2xh04f 26.11.2006 1303

Note that $f(x) \le 0$ if $x \in [-\sqrt{5}, \sqrt{5}\,]$ and $f(x) > 0$ if x is outside that interval. (See the figure – in problems like this it is usually a good idea to try to sketch the graph even if you are not asked to do so.) By the extreme value theorem, f has a global minimum point over $[-\sqrt{5}, \sqrt{5}]$, and it is clear that this point must be $x = -1$. It follows that $x = -1$ is a *global* minimum point for f over the entire real line, $\mathbb{R} = (-\infty, \infty)$. There is no global maximum point for f over $\mathbb{R}$, since $\lim_{x\to-\infty} f(x) = \infty$.

(c) Integration by parts yields

$$\begin{aligned}\int (x^2-5)e^{-x/2}\,dx &= (x^2-5)(-2e^{-x/2}) + \int 4xe^{-x/2}\,dx \\ &= -2(x^2-5)e^{-x/2} - 8xe^{-x/2} + 8\int e^{-x/2}\,dx \\ &= (-2x^2-8x+10)e^{-x/2} - 16e^{-x/2} + C \\ &= (-2x^2-8x-6)e^{-x/2} + C.\end{aligned}$$

It follows that

$$\int_0^b (x^2-5)e^{-x/2}\,dx = (-2b^2-8b-6)e^{-b/2} + 6e^0 \to 6 \text{ as } b\to\infty$$

because

$$\lim_{b\to\infty} b^p e^{-b/2} = \lim_{b\to\infty} \frac{b^2}{(\sqrt{e}\,)^b} = 0$$

for every constant p. (Cf. equation (4) on page 264 in EMEA, page 224 in MA I.) Alternatively, one can use l'Hôpital's rule to determine

$$\lim_{b\to\infty} \frac{-2b^2-8b-6}{e^{b/2}} = -\frac{\text{“}\infty\text{”}}{\infty} = \cdots .$$

Problem 2

(a) Using elementary operations, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 2 & b \end{vmatrix} \begin{matrix} -1 \\ \longleftarrow \\ \longleftarrow \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & a-1 \\ 0 & 1 & b-1 \end{vmatrix} \begin{matrix} \\ -1 \\ \longleftarrow \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & a-1 \\ 0 & 0 & b-a \end{vmatrix} = b-a.$$

Of course, we could also have used cofactor expansion along a row or column.

(b) The determinant of the equation system is precisely the determinant from part (a), so by Cramer's rule, the system has a unique solution if $a \ne b$.

If $a = b$, then the system becomes

$$(*) \qquad \begin{aligned} x + y + z &= c \\ x + 2y + az &= 2c \\ x + 2y + az &= 2 \end{aligned}$$

If $c \neq 1$, then the last two equations in $(*)$ contradict each other, and the system has no solutions. If $c = 1$, the $(*)$ reduces to

$$\begin{aligned} x + \ \ y + \ \ z &= 1 \\ x + 2y + az &= 2 \end{aligned} \iff \begin{aligned} x + y + \qquad\quad z &= 1 \\ y + (a-1)z &= 1 \end{aligned}$$

which has infinitely many solutions (with one degree of freedom). This is obvious, since for any value of z, the last equation will determine y, and then x is given by the first equation.

Conclusion: The system has

(i) a unique solution if $a \neq b$,

(ii) several solutions if $a = b$ and $c = 1$,

(iii) no solutions if $a = b$ and $c \neq 1$.

Problem 3

(a) With the Lagrangian

$$\mathcal{L}(x, y, z) = x + 2y + \ln(1+z) - \lambda(x^2 + y^2 - az),$$

the necessary Lagrange conditions for (x, y, z) to be a solution become

(1) $\qquad (\mathcal{L}'_x =) \qquad 1 - 2\lambda x = 0$

(2) $\qquad (\mathcal{L}'_y =) \qquad 2 - 2\lambda y = 0$

(3) $\qquad (\mathcal{L}'_z =) \qquad \dfrac{1}{1+z} + \lambda a = 0$

together with the constraint equation

(4) $\qquad x^2 + y^2 - az = 0$

(b) From conditions (2) and (1) we get $2\lambda y = 2 = 4\lambda x$. This shows that $\lambda \neq 0$, and so we get $y = 2x$. The constraint $x^2 + y^2 + 3z = 0$ then yields $3z = -x^2 - y^2 = -5x^2$, so $z = -\frac{5}{3}x^2$.

Conditions (3) and (1) now yield

$$\frac{1}{1+z} = -\lambda a = 3\lambda = \frac{3}{2x},$$

so

$$2x = 3(1+z) = 3 - 5x^2.$$

Hence, $5x^2 + 2x - 3 = 0$. This quadratic equation has the roots $x_1 = 3/5$ and $x_2 = -1$. The equations $y = 2x$ and $z = 5x^2/3$ then give the points $(x_1, y_1, z_1) = (3/5, 6/5, -3/5)$, $(x_2, y_2, z_2) = (-1, -2, -5/3)$ as the solutions of the first-order conditions. However, we must have $1 + z > 0$ for $f(x, y, z)$ to be defined, so z_2 is unusable.

M2xh04f 26.11.2006 1303

Given that there is a solution of the maximization problem, the solution must be

$$(x_1, y_1, z_1) = (3/5,\ 6/5,\ -3/5), \quad \text{with} \quad \lambda = 1/(2x_1) = 5/6.$$

The maximum value is $f_{\max} = x_1 + 2y_1 + \ln(1 + z_1) = 3 + \ln(2/5)$.

(With $\lambda = 5/6$, the Lagrangian becomes $\mathcal{L}(x, y, z) = x + 2y + \ln(1 + z) - \frac{5}{6}(x^2 + y^2 + 3z)$, which is concave. Hence, (x_1, y_1, z_1) certainly is a maximum point.)

(c) (i) If $a = 0$, the constraint becomes $x^2 + y^2 = 0$, which gives $x = y = 0$ without any restriction on z. We can then make $f(x, y, z) = f(0, 0, z) = \ln(1 + z)$ as large as we like, so there is no maximum.

(ii) With $a = 1$, the constraint gives $z = x^2 + y^2$, and so $f(x, y, z) = x + 2y + \ln(1 + x^2 + y^2)$, which can also be made arbitrarily large. So there is no maximum in this case either.

Problem 4

(a) For every matrix $\mathbf{C}$ we have $|\mathbf{C}^2| = |\mathbf{C}|^2 \geq 0$, whereas

$$|-\alpha\mathbf{I}_3| = \begin{vmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\alpha \end{vmatrix} = -\alpha^3 < 0.$$

Hence, there is no matrix $\mathbf{C}$ such that $\mathbf{C}^2 = -\alpha\mathbf{I}_3$.

(b) $(\mathbf{B} + \frac{1}{2}\mathbf{I}_3)^2 = \mathbf{B}^2 + \mathbf{B} + \frac{1}{4}\mathbf{I}_3$, so

$$\mathbf{B}^2 + \mathbf{B} + \mathbf{I}_3 = \mathbf{0} \iff (\mathbf{B} + \tfrac{1}{2}\mathbf{I}_3)^2 = -\tfrac{3}{4}\mathbf{I}_3.$$

According to part (a), this equation has no solutions.

M2xh04f 26.11.2006 1303