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# Answers to the examination problems in ECON3120/4120 Mathematics 2, 8 December 2006

## Problem 1

$f(x)$ is defined if and only if $\ln(x+2)$ is defined, i.e. if and only if $x+2>0$. Thus the domain of definition is $D_f = (-2, \infty)$.

Differentiation yields

$$f'(x) = \frac{1}{2} - \frac{1}{2}x + \frac{5}{x+2}\,, \qquad f''(x) = -\frac{1}{2} - \frac{5}{(x+2)^2}\,.$$

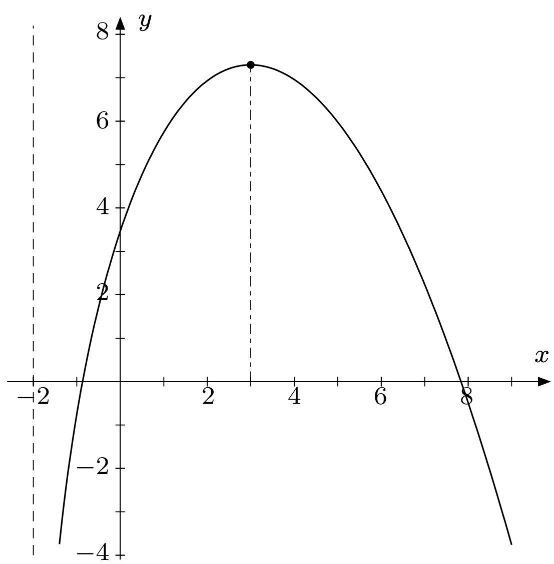
(b) The stationary points of $f$ are the solutions of the equation $f'(x) = 0$. We have

$$\frac{1}{2} - \frac{1}{2}x + \frac{5}{x+2} = 0 \iff \frac{5}{x+2} = \frac{x-1}{2} \iff 10 = (x+2)(x-1)$$
$$\iff x^2 + x - 12 = 0 \iff x = 3 \text{ or } x = -4.$$

The value $x = -4$ is outside the domain of $f$, so the only stationary point is $x = 3$.

Since $f''(x) < 0$ for all $x > -2$, the first derivative of $f$ is strictly decreasing. It follows that $f'(x) > 0$ for all $x$ in $(-2, 3)$ and $f'(x) < 0$ for all $x$ in $(3, \infty)$. Therefore $f$ is (strictly) increasing in $(-2, 3]$ and (strictly) decreasing in $[3, \infty)$. Hence $x = 3$ is a global maximum point for $f(x)$. The maximum value is $f(3) = 5\ln 5 - 3/4 \approx 7.29719$.

(c) Here is the graph of $f$. Note the vertical asymptote $x = -2$.




The graph of $f(x) = \frac{1}{2}x - \frac{1}{4}x^2 + 5\ln(x+2)$.



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The equation $f(x) = 0$ has exactly two solutions: The maximum value of $f$ is $f(3) = 5\ln(5) - 3/4 > 5 - 3/4 > 0$. On the other hand, it is clear that $\lim_{x\to -2^+} f(x) = -\infty$. It is also true that $\lim_{x\to\infty} f(x) = -\infty$. To see that, note that

$$f(x) = x^2\Big(\frac{1}{2x} - \frac{1}{4} + \frac{5\ln(x+2)}{x^2}\Big). \qquad (*)$$

By l'Hôpital's rule the fraction $5\ln(x+2)/x^2$ tends to 0 as $x \to \infty$, and therefore the whole expression in the big parenthesis in $(*)$ tends to $-\frac{1}{4}$.

By the intermediate value theorem ("skjæringssetningen"), there is a solution of the equation $f(x) = 0$ in each of the intervals $(-2, 3)$ and $(3, \infty)$. Since $f$ is strictly monotone in each of these intervals, there are no other solutions of the equation.

(For the curious: The two roots are $x_1 \approx -0.867547$ and $x_2 \approx 7.83515$.)

(d) By means of the substitution $u = x + 2$, $du = dx$, we get

$$\int_0^4 \Big(\frac{x}{2} - \frac{x^2}{4} + 5\ln(x+2)\Big)\,dx = \int_0^4 \Big(\frac{x}{2} - \frac{x^2}{4}\Big)\,dx + 5\int_2^6 \ln u\,du$$
$$= \Big|_0^4 \Big(\frac{x^2}{4} - \frac{x^3}{12}\Big) + 5\Big|_2^6 (u\ln u - u) = -\frac{64}{3} + 30\ln 6 - 10\ln 2.$$

**Problem 2**

(a) With the Lagrangian $\mathcal{L} = 24x - x^2 + 16y - 2y^2 - \lambda(x^2 + 2y^2 - 44)$ we get the first-order conditions

$$24 - 2x - 2\lambda x = 0 \iff 24 - 2(1+\lambda)x = 0, \qquad (1)$$
$$16 - 4y - 4\lambda y = 0 \iff 16 - 4(1+\lambda)y = 0, \qquad (2)$$

together with the constraint

$$x^2 + 2y^2 = 44\,. \qquad (3)$$

From (1) and (2) we get $(1+\lambda)x = 12$ and $(1+\lambda)y = 4$. Therefore $\lambda \neq -1$ and

$$x = \frac{12}{1+\lambda}\,, \qquad y = \frac{4}{1+\lambda}\,.$$

It follows that $x = 3y$. Using this in (3), we get $11y^2 = 44$, so $y = \pm 2$. The stationary points of the Lagrangian are therefore

$$(x_1, y_1) = (6, 2) \text{ with } \lambda_1 = 1, \qquad (x_2, y_2) = (-6, -2) \text{ with } \lambda_2 = -3.$$

Inserting these values of $x$ and $y$ into $f(x, y) = 24x - x^2 + 16y - 2y^2$ yields

$$f(6, 2) = 132, \qquad f(-6, -2) = -220.$$

The maximum point is therefore $(6, 2)$.

(We know that there is a maximum point because $f$ is continuous and the curve $x^2 + 2y^2 = 44$ is a closed and bounded set. Also, the gradient of $g(x, y) =$



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$x^2 + 2y^2$ is never zero along the constraint curve, and therefore every extreme point in our problem must satisfy the Lagrange conditions. See Theorem 14.3.1 in EMEA or Setning 14.3.1 in MA I.)

**Problem 3**

(a) Cofactor expansion along the first column yields

$$|\mathbf{A}_t| = 1\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - (-t)\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + t\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 4 - 0 + t(-4) = 4(1-t).$$

In particular, $|\mathbf{A}_2| = -4$, and therefore $|(\mathbf{A}_2)^3| = |\mathbf{A}_2|^3 = (-4)^3 = -64$.

It is also possible to find $|(\mathbf{A}_2)^3|$ by first calculating $(\mathbf{A}_2)^3$ and then taking the determinant. This is not a very good idea, since it involves a lot of work and a corresponding risk of mistakes in calculation, but if really want to do it that way, here are the matrices you will get:

$$(\mathbf{A}_2)^2 = \mathbf{A}_2\mathbf{A}_2 = \begin{pmatrix} 3 & 6 & 8 \\ -4 & 9 & 6 \\ 4 & 7 & 10 \end{pmatrix}, \quad (\mathbf{A}_2)^3 = (\mathbf{A}_2)^2\mathbf{A}_2 = \begin{pmatrix} 7 & 29 & 34 \\ -10 & 29 & 22 \\ 10 & 35 & 42 \end{pmatrix}.$$

(b) Matrix multiplication yields

$$\mathbf{A}_2\mathbf{B} = \begin{pmatrix} 1 & 0 & s - \frac{3}{2} \\ 0 & 1 & 3s - \frac{9}{2} \\ 0 & 0 & s - \frac{1}{2} \end{pmatrix}.$$

For $s = 3/2$ this matrix equals $\mathbf{I}_3$, and so for this value of $s$,

$$\mathbf{B} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 1/2 & 3/2 \\ 2 & -1/4 & -5/4 \end{pmatrix} = (\mathbf{A}_2)^{-1}.$$

**Problem 4**

Differentiating (i.e. taking derivatives) with respect to $t$, and remembering that $x$ and $y$ are functions of $t$, we get

$$x^2 + t\,2x\dot{x} + \dot{y} = 2,$$
$$\frac{2}{x}\dot{x} + 3\dot{y} = \dot{x} + \frac{1}{y}\dot{y} + 1.$$

At the point $(x, y, t) = (1, \frac{1}{2}, 1)$ we get

$$\begin{aligned} 1 + 2\dot{x} + \dot{y} &= 2 \\ 2\dot{x} + 3\dot{y} &= \dot{x} + 2\dot{y} + 1 \end{aligned} \quad \iff \quad \begin{aligned} 2\dot{x} + \dot{y} &= 1 \\ \dot{x} + \dot{y} &= 1 \end{aligned}$$

with the solution $\dot{x} = dx/dt = 0$, $\dot{y} = dy/dt = 1$.