Department of Economics
A. Strøm

Answers to the examination problems in ECON 3120/4120, 30 May 2005

Problem 1

(a) The first and second order derivatives are

$$f'(x) = (\ln x)^2 + x \cdot 2\ln x \cdot \frac{1}{x} = (\ln x)^2 + 2\ln x = (\ln x + 2)\ln x\,,$$
$$f''(x) = 2\ln x \cdot \frac{1}{x} + \frac{2}{x} = \frac{2}{x}(\ln x + 1)\,.$$

(b) To determine the sign of $f'(x)$, it is best to use the last version of $f'(x)$ given in part (a). The factor $\ln x + 2$ changes sign at $x = e^{-2}$ (because that's where $\ln x = -2$), and $\ln x$ changes sign at $x = 1$. A sign diagram easily shows that

$$f'(x) \begin{cases} > 0 & \text{if } 0 < x < e^{-2}\,, \\ < 0 & \text{if } e^{-2} < x < 1\,, \\ > 0 & \text{if } x > 1\,. \end{cases}$$

It follows that f is (strictly) increasing in $(0, e^{-2}]$, (strictly) decreasing in $[e^{-2}, 1]$, and (strictly) increasing in $[1, \infty)$.

It is clear that $f(1) = 0$ and that $f(x) > 0$ for all positive values of x different from 1 (because then $x > 0$ and $\ln x \neq 0$). Therefore $x = 1$ is a global minimum point for f and it is the only global minimum point. The function has no global maximum point because $f(x) \to \infty$ as $x \to \infty$.

(c) By l'Hôpital's rule,

$$\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{1/x} = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to 0^+} \frac{(2\ln x)/x}{-1/x^2}$$
$$= \lim_{x\to 0^+} \frac{2\ln x}{-1/x} = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to 0^+} \frac{2/x}{1/x^2} = \lim_{x\to 0^+} 2x = 0\,.$$

The second row here will be simplified if you recall that $\lim_{x\to 0^+} x\ln x = 0$.

A more efficient way to find $\lim_{x\to 0^+} x(\ln x)^2$ is the following. Let $u = -\ln x$. Then $x = e^{-u}$, so $u \to \infty \iff x \to 0^+$, and

$$\lim_{x\to 0^+} x(\ln x)^2 = \lim_{u\to\infty} e^{-u}u^2 = \lim_{u\to\infty} \frac{u^2}{e^u} = 0\,.$$

(See formula (7.12.3) on p. 265 in EMEA or (6.5.4) on p. 224 in MA II.)

To find $\lim_{x\to 0^+} f'(x)$, we shall use the product form of the derivative, $f'(x) = (\ln x + 2)\ln x$. Since $\ln x \to -\infty$ as $x \to 0^+$, both factors in the product tend to $-\infty$, and therefore $\lim_{x\to 0^+} f'(x) = \infty$.

m2xv05fe 22.9.2008 23

Several candidates tried the following *incorrect argument:* Since $f'(x) = (\ln x)^2 + 2\ln x$, where $(\ln x)^2 \to \infty$ and $\ln x \to -\infty$, the limit of $f'(x)$ must be $\infty - \infty = 0$ (or even $\infty - 2\infty = -\infty$). This kind of argument is nonsense, for if you have two expressions that both tend to ∞, there is no general rule about what happens to their difference. We do know that the *sum* will tend to ∞, but we do not know about the difference. Remember: $\infty - \infty$ *is undefined.*

Problem 2

(a) We use l'Hôpital's rule twice:

$$\lim_{x\to 0} \frac{e^{xt} - 1 - xt}{x^2} = \frac{\text{“}0\text{”}}{0} = \lim_{x\to 0} \frac{te^{xt} - t}{2x} = \frac{\text{“}0\text{”}}{0} = \lim_{x\to 0} \frac{t^2 e^{xt}}{2} = \frac{t^2}{2}\,.$$

Note that the differentiations are done with respect to x.

(b) Introducing $u = e^{2x} + 1$ as a new variable, we get $du = 2e^{2x}\,dx = 2(u-1)\,du$, so $dx = \frac{du}{2(u-1)}$. Also, $e^{4x} = (e^{2x})^2 = (u-1)^2$. Therefore

$$\begin{aligned}\int \frac{e^{4x}}{e^{2x}+1}\,dx &= \int \frac{(u-1)^2}{u} \frac{du}{2(u-1)} = \frac{1}{2}\int \frac{u-1}{u}\,du = \frac{1}{2}\int \Big(1 - \frac{1}{u}\Big)du \\ &= \frac{1}{2}(u - \ln u) + C = \frac{1}{2}\big(e^{2x} + 1 - \ln(e^{2x}+1)\big) + C \\ &= \frac{1}{2}e^{2x} - \frac{1}{2}\ln(e^{2x}+1) + C_1,\end{aligned}$$

with $C_1 = C + \frac{1}{2}$.

(c) Integration by parts yields

$$\begin{aligned}\int (\ln x)^2\,dx = \int (\ln x)^2 \cdot 1\,dx &= (\ln x)^2 \cdot x - \int 2(\ln x)\frac{1}{x} \cdot x\,dx \\ &= x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2\int (\ln x)\cdot 1\,dx \\ &= x(\ln x)^2 - 2\Big((\ln x)\cdot x - \int \frac{1}{x}\cdot x\,dx\Big) \qquad \text{(by parts again)} \\ &= x(\ln x)^2 - 2x\ln x + 2\int 1\,dx \\ &= x(\ln x)^2 - 2x\ln x + 2x + C.\end{aligned}$$

Problem 3

(a) Cofactor expansion along the first row yields

$$|\mathbf{A}_t| = \begin{vmatrix} 0 & t & 1 \\ 4 & -2 & 8 \\ 1 & 1 & 1 \end{vmatrix} = 0\cdot(\cdots) - t\begin{vmatrix} 4 & 8 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix} = 4t + 6.$$

(b) Carrying out the matrix multiplications on the left side of the equation, we get

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & 0 \end{pmatrix} - \begin{pmatrix} x & y \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2x+z & 2y \\ -x & -y \end{pmatrix} - \begin{pmatrix} 2y & x+y \\ 0 & z \end{pmatrix}$$
$$= \begin{pmatrix} 2x-2y+z & -x+y \\ -x & -y-z \end{pmatrix}.$$

The original matrix equation therefore leads to the four equations

$$\begin{array}{llll} (1) & 2x-2y+z=5 & (3) & -x+y=-2 \\ (2) & -x=0 & (4) & -y-z=1 \end{array}$$

From (2) we get $x = 0$ and then (3) yields $y = -2$. Equation (1) gives $z = 5+2y = 1$, and then equation (4) is also satisfied. Thus the problem has the unique solution

$$x = 0, \quad y = -2, \quad z = 1.$$

Note that (1)–(4) is a system of four equations in three unknowns. We only needed three of the equations to find x, y, and z, but we also had to check that the values we found satisfied the fourth equation as well.

Problem 4

(a) The Lagrangian is

$$\mathcal{L}(x, y, a) = x^2 + y^2 + z - \lambda(x^2 + 2xy + y^2 + x^2 - a) - \mu(x + y + z - 1)$$

where λ and μ are the Lagrange multipliers. The necessary first-order conditions for (x, y, z) to be a minimum point are:

$$\begin{array}{lr} (1) & \mathcal{L}'_1(x,y,z) = 2x - 2\lambda(x+y) - \mu = 0 \\ (2) & \mathcal{L}'_1(x,y,z) = 2y - 2\lambda(x+y) - \mu = 0 \\ (3) & \mathcal{L}'_1(x,y,z) = \qquad 1 - 2\lambda z - \mu = 0 \end{array}$$

together with the constraints

$$\begin{array}{lr} (4) & x^2 + 2xy + y^2 + z^2 = a \\ (5) & x + y + z = 1 \end{array}$$

From (1) and (2) we get $2x = 2\lambda(x + y) + \mu = 2y$, so $x = y$. It then follows from (5) that $z = 1 - 2x$, and (4) now gives

$$x^2 + 2x^2 + x^2 + (1 - 2x)^2 = 5/2\,.$$

The roots of this quadratic equation are $x_1 = 3/4$ and $x_2 = -1/4$. Hence there are two points that satisfy the Lagrange conditions:

$$(x_1, y_1, z_1) = (3/4, 3/4, -1/2)\,, \qquad (x_2, y_2, z_2) = (-1/4, -1/4, 3/2)\,.$$

To find the corresponding values of λ and μ, we can use equations (1) and (3). The results are

$$(\lambda_1, \mu_1) = (1/8, 9/8)\,, \qquad (\lambda_2, \mu_2) = (3/8, -1/8)\,.$$

Given that there is a minimum point in the problem, we just have to check the value of $f(x, y, z) = x^2 + y^2 + z$ at the two points we have found. We find

$$f(x_1, y_1, z_1) = 5/8, \qquad f(x_2, y_2, z_2) = 13/8.$$

Thus the minimum point is (x_1, y_1, z_1).

Warning: Do not fall into the trap of thinking that (x_2, y_2, z_2) must be a maximum point just because it is the only other stationary point of the Lagrangian. In fact, there is no maximum point, because the point $(x, y, z) = (t - 1/4, -t - 1/4, 3/2)$ satisfies the constraints for all t, and $f(t-1/4, -t-1/4, 3/2) = 2t^2 + 1/8$ can be made as large as we like by choosing suitable values of t.

How can anyone dream up points like that? Well, note that equations (4) and (5) can be written as

$$(x + y)^2 + z^2 = a, \qquad (x + y) + z = 1\,,$$

so they actually place restrictions only on $x + y$ and z, namely

$$x + y = \frac{1 \pm \sqrt{2a - 1}}{2}\,, \qquad z = \frac{1 \mp \sqrt{2a - 1}}{2}\,.$$

Don't worry, we wouldn't expect you to do anything like this on the exam. Just don't take it for granted that a stationary point that is not a minimum point must automatically be a maximum point.

(c) $V'(5/2) = \lambda_1 = 1/8$.