Department of Economics
A. Strøm

Answers to the examination problems in ECON3120/4120 Mathematics 2, 4 June 2007

Problem 1

(a) Cofactor expansion along the first row gives

$$|\mathbf{A}_a| = 3\begin{vmatrix} 1 & 2a-3 \\ a & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2a-3 \\ 2 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 1 & 1 \\ 2 & a \end{vmatrix}$$
$$= 3(2-2a^2+3a) - 2(8-4a) - 4(a-2) = -6a^2+13a-2$$

(b) The coefficient matrix of the system is precisely the matrix $\mathbf{A}_a$ from part (a). Cramer's rule tells us that the system has a unique solution if and only if $|\mathbf{A}_a| \neq 0$. Now, by the usual formula for solving quadratic equations,

$$|\mathbf{A}_a| = 0 \iff 6a^2 - 13a + 2 = 0 \iff a = \frac{13 \pm \sqrt{13^2 - 4\cdot 6\cdot 12}}{12} = \frac{13 \pm 11}{12}$$
$$\iff a = 2 \text{ or } a = 1/6.$$

Thus, for all values of a except 2 and 1/6, the system has a unique solution.

If $a = 2$, the system becomes

$$\begin{aligned} 3x + 2y - 4z &= 2 \\ x + y + z &= 3 \\ 2x + 2y + 2z &= 6 \end{aligned} \iff \begin{aligned} 3x + 2y - 4z &= 2 \\ x + y + z &= 3 \end{aligned} \iff \begin{aligned} 3x + 2y &= 2 + 4z \\ x + y &= 3 - z \end{aligned}$$

The second and third equations on the left are obviously equivalent, so we can drop one of them. In the final system, we can choose any value we like for z, and then x and y are uniquely determined (by Cramer's rule, if you like).

Finally, for $a = 1/6$, the system becomes

$$\begin{aligned} 3x + 2y - 4z &= 2 \\ x + y - \tfrac{8}{3}z &= 3 \quad -3 \quad -2 \\ 2x + \tfrac{1}{6}y + 2z &= 6 \end{aligned}$$

The elementary operations indicated lead to

$$\begin{aligned} -y + 4z &= -7 \\ x + y - \tfrac{8}{3}z &= 3 \\ -\tfrac{11}{6}y + \tfrac{22}{3}z &= 0 \quad \times \tfrac{6}{11} \end{aligned} \quad \sim \quad \begin{aligned} -y + 4z &= -7 \\ x + y - \tfrac{8}{3}z &= 3 \\ -y + 4z &= 0 \end{aligned}$$

The final system is obviously inconsistent, since the first and last equations contradict each other. Hence, the original system has no solutions for $a = 1/6$.

m2xv07fasit 10.6.2007 921

Problem 2

The given equation is a linear differential equation of the form $\dot{x} + ax = b(t)$, with $a = -1$ and $b(t) = e^t/t$. It can be solved by formula (5.4.4) on page 199 in FMEA (formula (1.4.5) on page 13 in MA II). The formula gives

$$x = Ce^{-at} + e^{-at}\int e^{at}b(t)\,dt = Ce^t + e^t\int \frac{1}{t}\,dt = Ce^t + e^t \ln t = e^t(C + \ln t).$$

Of course, we could also have used the general formula (5.4.6) on page 200 ((1.4.6) on page 15 in MA II) with $a(t) = -1$ and $A(t) = -t$.

The solution passes through $(t, x) = (1, e^{-1})$ if C is such that

$$e^1(C + \ln 1) = e^{-1} \iff eC = e^{-1} \iff C = e^{-2}.$$

Problem 3

(a) We get

$$\begin{aligned} dx + e^{v-u}(dv - du) - \tfrac{1}{y}\,dy &= 0 \\ y\,dx + x\,dy - du + 4v\,dv &= 0 \end{aligned}$$

(b) Write the equations from part (a) as a linear equation system with du and dv as the unknowns:

$$\begin{aligned} -e^{v-u}\,du + e^{v-u}\,dv &= -dx + \tfrac{1}{y}\,dy \\ -du + \quad 4v\,dv &= -y\,dx - x\,dy \end{aligned} \iff \begin{aligned} -du + \quad dv &= -e^{u-v}\,dx + \tfrac{e^{u-v}}{y}\,dy \\ -du + 4v\,dv &= -y\,dx - x\,dy \end{aligned}$$

Subtracting the second equation from the first gives

$$(1 - 4v)\,dv = (y - e^{u-v})\,dx + \frac{e^{u-v} + xy}{y}\,dy\,,$$

so

$$dv = \frac{y - e^{u-v}}{1 - 4v}\,dx + \frac{e^{u-v} + xy}{y(1 - 4v)}\,dy\,.$$

It follows that

$$v'_y = \frac{\partial v}{\partial y} = \frac{e^{u-v} + xy}{y(1 - 4v)}\,.$$

Problem 4

(a) Let $F(x, y, u) = u + \ln u - Ax - \frac{1}{2}y^2$. Then

$$u'_x = -\frac{F'_1(x, y, u)}{F'_3(x, y, u)} = \frac{A}{1 + 1/u} = \frac{Au}{u + 1}\,,$$

and

$$u'_y = -\frac{F'_2(x, y, u)}{F'_3(x, y, u)} = -\frac{-y}{1 + 1/u} = \frac{yu}{u + 1}\,.$$

(b) The Lagrangian for problem (P) is

$$\mathcal{L}(x, y) = ax + by - \lambda(u(x, y) - K)\,,$$

and the first-order conditions become

$$\text{(1)} \qquad \mathcal{L}'_x(x, y) = a - \lambda u'_x = 0 \iff a - \frac{\lambda A u}{u+1} = 0$$

$$\text{(2)} \qquad \mathcal{L}'_y(x, y) = b - \lambda u'_y = 0 \iff b - \frac{\lambda y u}{u+1} = 0$$

together with the constraint

$$\text{(3)} \qquad u(x, y) = K \iff Ax + \tfrac{1}{2}y^2 = K + \ln K\,.$$

Equation (1) yields

$$\lambda = \frac{a}{u'_x} = \frac{a(u+1)}{Au}\,,$$

and then (2) gives

$$y = \frac{b(u+1)}{\lambda u} = \frac{bA}{a}\,.$$

The value of x is then determined from (3), and we have found that the first-order conditions have the unique solution

$$(x^*, y^*) = \Big(\frac{K + \ln K}{A} - \frac{b^2 A}{2a^2}\,,\, \frac{bA}{a}\Big)$$

(c) We have

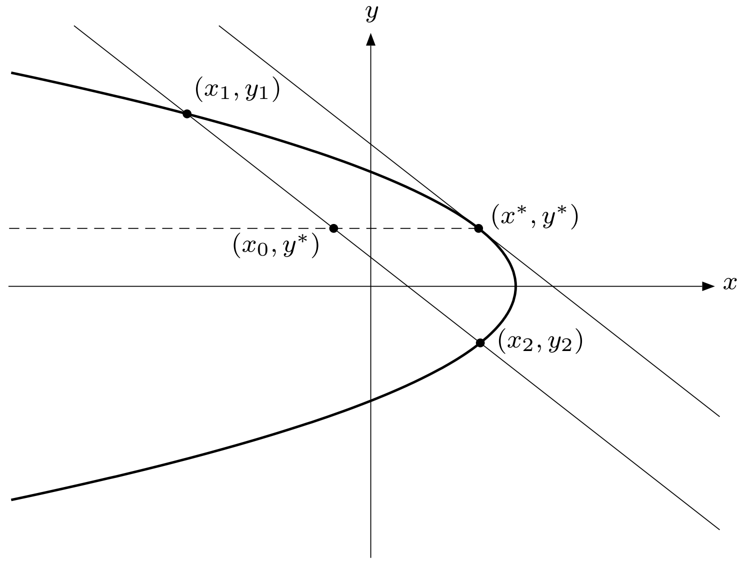
$$\begin{aligned} u(x, y) = K &\iff u(x, y) + \ln u(x, y) = K + \ln K \\ &\iff Ax + \frac{1}{2}y^2 = K + \ln K \\ &\iff y^2 = Q - 2Ax \iff y = \pm\sqrt{Q - 2Ax}\,, \end{aligned}$$

where $Q = 2(K + \ln K)$.

(d) The result in part (c) shows that the level curve $u(x, y) = K$ is a parabola with a horizontal axis and opening towards the left. The figure shows this parabola for one value of K together with a couple of level curves of $f(x, y) = ax + by$ for an arbitrary choice of values for a and b. For a given choice of a and b, all level curves of f are straight lines and they are all parallell. It is clear that the point (x^*, y^*) lies on the rightmost of all those level curves that have at least one point in common with the parabola.

To decide whether (x^*, y^*) is a maximum or a minimum point of $f(x, y) = ax+by$ we compare the value at this point with the value at the other points on the parabola, like (x_1, y_1) in the figure. The level curve through this point intersects the horizontal line $y = y^*$ at a point (x_0, y^*), and we get

$$f(x^*, y^*) - f(x_1, y_1) = f(x^*, y^*) - f(x_0, y^*) = a(x^* - x_0).$$



For problem 4(d)

It is clear that $x^* > x_0$, and it follows that (x^*, y^*) is a maximum point in problem (P) if $a > 0$ (and a minimum point if $a < 0$). The same argument works equally well for points lying below the line $y = y^*$, like (x_2, y_2) in the figure.

Note that the sign of b does not matter. If $b = 0$, then $f(x, y) = ax$ and the level curves of f are vertical straight lines. If $b \neq 0$, then the level curves of f have a negative slope if b has the same sign as a, and a positive slope if b has the opposite sign. Also note that $(-a)x + (-b)y = -(ax + by)$ has the same level curves as $ax + by$ (but corresponding to different function values).