University of Oslo
Department of Economics

English

ECON3120/4120 Mathematics 2

Wednesday 2 June 2010, 09:00–12:00.

There are 2 pages of problems to be solved.

All printed and written material may be used. Pocket calculators are allowed.

State reasons for all your answers.

Grades given run from A (best) to E for passes, and F for fail.

Problem 1

Let f be the function given by $f(x, y) = \ln(1 + x) + 3\ln(1 + y)$.

(a) Use Lagrange's method to solve the problem

$$\text{maximize} \quad f(x, y) \quad \text{subject to} \quad ax + y = m\,,$$

where a and m are positive constants. You may take it as given that the problem has a solution.

(b) Consider the triangular region $T = \{(x, y) : x \geq 0, y \geq 0, 2x + y \leq 8\}$ in the xy-plane. Show that $f(x, y)$ attains both a maximum and a minimum over T, and find the maximum and minimum points.

Problem 2

(a) Lat f be a differentiable function with $f(x) > 0$ for all x. Show that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C\,.$$

(b) Find the general solution of the separable differential equation

$$(1 + t^2)e^x \dot{x} = 2t(1 + e^x)\,. \qquad (*)$$

(c) One of the solution curves for the equation $(*)$ passes through the point $(t_0, x_0) = (1, 0)$. Show that the tangent to that curve at this point intersects the line $x = t$ at $(2, 2)$.

(Cont.)

Problem 3

A market survey for a certain good showed that the sales at time t would be given by

$$S(t) = Ce^{-at}\left(e^{-at} + b\right)^{-2},$$

where a, b, and C are positive constants.

(a) Find an expression for $S'(t)$.

(b) Show that S has exactly one stationary point t^*, and show that t^* maximizes $S(t)$ over $(-\infty, \infty)$.

(c) Find a necessary and sufficient condition for $t^* > 0$.

(d) Show that the integral $\int_0^\infty S(t)\,dt$ converges, and find its value.

Problem 4

For each real number t we define the matrix $\mathbf{A}_t$ by

$$\mathbf{A}_t = \begin{pmatrix} 0 & 1 & t \\ 1 & 0 & -t \\ t-1 & 1 & 1 \end{pmatrix}$$

(a) Calculate the determinant $|\mathbf{A}_t|$.

(b) Find all solutions of the system of linear equations

$$\begin{aligned} y + tz &= 0 \\ x \qquad - tz &= 0 \\ (t-1)x + y + \ z &= 0 \end{aligned}$$

(i) for $t = 1$; (ii) for $t = 2$.

(c) If $\mathbf{B}$ and $\mathbf{C}$ are $n \times n$ matrices we say that they *commute* with each other if $\mathbf{BC} = \mathbf{CB}$. Show that if $\mathbf{B}$ and $\mathbf{C}$ commute with each other and $\mathbf{B}$ has an inverse, then $\mathbf{B}^{-1}$ and $\mathbf{C}$ will also commute with each other.

Problem 5

Let $f(x) = x^{1/\ln(e^x - 1)}$ for all $x > 0$, $x \neq \ln 2$. Find the limit $\lim_{x \to 0^+} f(x)$ if it exists. (*Hint:* Look at $\ln f(x)$.)