Student's Manual

Further Mathematics for Economic Analysis

2nd edition

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Preface

This student's solutions manual accompanies *Further Mathematics for Economic Analysis* (2nd edition, FT Prentice Hall, 2008). Its main purpose is to provide more detailed solutions to the problems marked with Ⓢⓜ in the text. The Manual should be used in conjunction with the answers in the book. In some few cases only a part of the problem is done in detail, because the rest follows the same pattern.

At the end of this manual there is a list of misprints and other errors in the book, and even one or two in the errata list in the preliminary and incomplete version of this manual released in September this year. We would greatly appreciate suggestions for improvements from our readers as well as help in weeding out the inaccuracies and errors that probably remain.

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Chapter 1 Topics in Linear Algebra

1.2

1.2.3 Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ be a matrix with the three given vectors as columns. Cofactor expansion of $|\mathbf{A}|$ along the first row yields

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0 = 1 - 2(-1) = 3 \neq 0$$

By Theorem 1.2.1 this shows that the given vectors are linearly independent.

1.2.6 Part (a) is just a special case of part (b), so we will only prove (b). To show that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are linearly independent it suffices to show that if $c_1, c_2, \dots, c_n$ are real numbers such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

then all the c_i have to be zero. So suppose that we have such a set of real numbers. Then for each $i = 1, 2, \dots, n$, we have

$$\mathbf{a}_i \cdot (c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n) = \mathbf{a}_i \cdot \mathbf{0} = 0 \tag{1}$$

Since $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ when $i \neq j$, the left-hand side of (1) reduces to $\mathbf{a}_i \cdot (c_i \mathbf{a}_i) = c_i \|\mathbf{a}_i\|^2$. Hence, $c_i \|\mathbf{a}_i\|^2 = 0$. Because $\mathbf{a}_i \neq \mathbf{0}$ we have $\|\mathbf{a}_i\| \neq 0$, and it follows that $c_i = 0$.

1.3

1.3.1 (a) The rank is 1. See the answer in the book.

(b) The minor formed from the first two columns in the matrix is $\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6 \neq 0$. Since this minor is of order 2, the rank of the matrix must be at least 2, and since the matrix has only two rows, the rank cannot be greater than 2, so the rank equals 2.

(c) The first two rows and last two columns of the matrix yield the minor $\begin{vmatrix} -1 & 3 \\ -4 & 7 \end{vmatrix} = 5 \neq 0$, so the rank of the matrix is at least 2. On the other hand, all the four minors of order 3 are zero, so the rank is less than 3. Hence the rank is 2. (It can be shown that $\mathbf{r}_2 = 3\mathbf{r}_1 + \mathbf{r}_3$, where $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are the rows of the matrix.)

An alternative argument runs as follows: The rank of a matrix does not change if we add a multiple of one row to another row, so

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{pmatrix} \begin{matrix} -2 & 1 \\ \leftarrow\!\lrcorner & \\ \longleftarrow & \end{matrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

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Here $\sim$ means that the last matrix is obtained from the first one by elementary row operations. The last matrix obviously has rank 2, and therefore the original matrix also has rank 2.

(d) The first three columns of the matrix yield the minor $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 1 & -1 & 2 \end{vmatrix} = -4 \neq 0$, so the rank is 3.

(e) $\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} = 9 \neq 0$, so the rank is at least 2. All the four minors of order 3 are zero, so the rank must be less than 3. Hence the rank is 2. (The three rows, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, of the matrix are linearly dependent, because $\mathbf{r}_2 = -14\mathbf{r}_1 + 9\mathbf{r}_3$.)

(f) The determinant of the whole matrix is zero, so the rank must be less than 4. On the other hand, the first three rows and the first three columns yield the minor

$$\begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = -7 \neq 0$$

so the rank is at least 3. It follows that the matrix has rank 3.

1.3.2 (a) The determinant is $(x+1)(x-2)$. The rank is 3 if $x \neq -1$ and $x \neq 2$. The rank is 2 if $x = -1$ or $x = 2$.

(a) By cofactor expansion along the first row, the determinant of the matrix $\mathbf{A} = \begin{pmatrix} x & 0 & x^2-2 \\ 0 & 1 & 1 \\ -1 & x & x-1 \end{pmatrix}$ is

$$|\mathbf{A}| = x \cdot (-1) - 0 \cdot 1 + (x^2 - 2) \cdot 1 = x^2 - x - 2 = (x+1)(x-2)$$

If $x \neq -1$ and $x \neq 2$, then $|\mathbf{A}| \neq 0$, so the rank of $\mathbf{A}$ equals 3. If $x = -1$ or $x = 2$, then $|\mathbf{A}| = 0$ and $r(\mathbf{A}) \leq 2$. On the other hand, the minor we get if we strike out the first row and the third column in $\mathbf{A}$ is $\begin{vmatrix} 0 & 1 \\ -1 & x \end{vmatrix} = 1 \neq 0$ for *all* x, so $r(\mathbf{A})$ can never be less than 2.

Conclusion: $r(\mathbf{A}) = \begin{cases} 2 & \text{if } x = -1 \text{ or } x = 2 \\ 3 & \text{otherwise} \end{cases}$

(b) A little calculation shows that the determinant of the matrix is $t^3 + 4t^2 - 4t - 16$, and if we note that this expression has $t + 4$ as a factor, it follows that the determinant is

$$t^3 + 4t^2 - 4t - 16 = t^2(t+4) - 4(t+4) = (t^2 - 4)(t+4) = (t+2)(t-2)(t+4)$$

Thus, if t does not equal any of the numbers -2, 2, and -4, the rank of the matrix is 3.

If we strike out the second row and the first column of the matrix, we get the minor $\begin{vmatrix} 5 & 6 \\ 1 & t+4 \end{vmatrix} = 5t + 14$, which is different from 0 for all the three special values of t that we found above, and thus the rank of the matrix is

$$\begin{cases} 2 & \text{if } t = -4, -2, \text{ or } 2 \\ 3 & \text{otherwise} \end{cases}$$

(c) The first and third rows are identical, as are the second and fourth. But the first two rows are always linearly independent. So the rank is 2 for all values of x, y, z, and w.

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1.4

1.4.2 (a) It is clear that $x_1 = x_2 = x_3 = 0$ is a solution. The determinant of the coefficient matrix is

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 9 \neq 0$$

Hence, by Cramer's rule the solution is unique and the system has 0 degrees of freedom. (This agrees with Theorem 1.4.2: since the rank of the coefficient matrix is 3 and there are 3 unknowns, the system has $3 - 3 = 0$ degrees of freedom.)

(b) By Gaussian elimination (or other means) we find that the solution is $x_1 = a$, $x_2 = -a$, $x_3 = -a$, and $x_4 = a$, with a arbitrary. Thus, there is one degree of freedom. (We could get the number of degrees of freedom from Theorem 1.4.2 in this case, too. The minor formed from the first three columns of the coefficient matrix has determinant $-3 \neq 0$, so the rank of the coefficient matrix is 3. Since there are 4 unknowns, the system has $4 - 3 = 1$ degree of freedom.)

1.4.3 The determinant of the coefficient matrix is $a^2 - 7a = a(a-7)$. Thus, if $a \neq 0$ and $a \neq 7$, the system has a unique solution. If $a = 0$ or $a = 7$, the rank of the coefficient matrix is 2 (why?), and the system either has solutions with 1 degree of freedom or has no solutions at all, depending on the value of b. One way of finding out is by Gaussian elimination, i.e. by using elementary row operations on the augmented coefficient matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & a & -21 & 2 \\ 3 & 7 & a & b \end{pmatrix} \begin{matrix} 1 & -3 \\ \leftarrow & \\ \leftarrow & \end{matrix} \sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & a+2 & -18 & 3 \\ 0 & 1 & a-9 & b-3 \end{pmatrix} \begin{matrix} \leftarrow \\ -(a+2) \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -a^2+7a & -ab-2b+3a+9 \\ 0 & 1 & a-9 & b-3 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & a-9 & b-3 \\ 0 & 0 & a(7-a) & -ab-2b+3a+9 \end{pmatrix}$$

This confirms that as long as $a \neq 0$ and $a \neq 7$, the system has a unique solution for any value of b. But if $a = 0$ or $a = 7$, the system has solutions if and only if $-ab - 2b + 3a + 9 = 0$, and then it has solutions with 1 degree of freedom. (The rank of the coefficient matrix is 2 and there are 3 unknowns.)

If $a = 0$, then the system has solutions if and only if $b = 9/2$.

If $a = 7$, then the system has solutions if and only if $-9b + 30 = 0$, i.e. $b = 10/3$.

1.4.6 (a) $|\mathbf{A}_t| = (t-2)(t+3)$, so $r(\mathbf{A}_t) = 3$ if $t \neq 2$ and $t \neq -3$. Because the upper left 2×2 minor of $\mathbf{A}_t$ is $-1 \neq 0$, the rank of $\mathbf{A}_t$ can never be less than 2, so $r(\mathbf{A}_2) = 2$, $r(\mathbf{A}_{-3}) = 2$.

(b) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. The vector equation $\mathbf{A}_{-3}\mathbf{x} = \begin{pmatrix} 11 \\ 3 \\ 6 \end{pmatrix}$ is equivalent to the equation system

$$x_1 + 3x_2 + 2x_3 = 11 \qquad (1)$$

$$2x_1 + 5x_2 - 3x_3 = 3 \qquad (2)$$

$$4x_1 + 10x_2 - 6x_3 = 6 \qquad (3)$$

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Equation (3) is obviously equivalent to (2), so we can remove it, and then we are left with the two equations

$$x_1 + 3x_2 + 2x_3 = 11 \qquad (1)$$
$$2x_1 + 5x_2 - 3x_3 = 3 \qquad (2)$$

We can consider these equations as an equation system with x_1 and x_2 as the unknowns:

$$x_1 + 3x_2 = 11 - 2x_3 \qquad (1')$$
$$2x_1 + 5x_2 = 3 + 3x_3 \qquad (2')$$

For each value of x_3 this system has the unique solution $x_1 = 19x_3 - 46$, $x_2 = -7x_3 + 19$. Thus the vector equation $\mathbf{A}_3\mathbf{x} = (11, 3, 6)'$ has the solution

$$\mathbf{x} = (19s - 46,\ -7s + 19,\ s)'$$

where s runs through all real numbers.

1.5

1.5.1 For convenience, let **A**, **B**, ..., **F** denote the matrices given in (a), (b), ..., (f), respectively.

(a) The characteristic polynomial of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$ is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 5 = (\lambda + 1)(\lambda + 5)$$

so the eigenvalues of **A** are $\lambda_1 = -1$ and $\lambda_2 = -5$. The eigenvectors corresponding to an eigenvalue λ are the vectors $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \neq \mathbf{0}$ that satisfy $\mathbf{Ax} = \lambda\mathbf{x}$, i.e.

$$\left.\begin{aligned} 2x - 7y &= -x \\ 3x - 8y &= -y \end{aligned}\right\} \iff 3x = 7y \quad \text{for } \lambda = -1$$

and

$$\left.\begin{aligned} 2x - 7y &= -5x \\ 3x - 8y &= -5y \end{aligned}\right\} \iff 7x = 7y \quad \text{for } \lambda = -5$$

This gives us the eigenvectors $\mathbf{v}_1 = s\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where s and t are arbitrary real numbers (different from 0).

(b) The characteristic equation of **B** is $|\mathbf{B} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 4 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 20 = 0$. **B** has two complex eigenvalues, $4 \pm 2i$, and no real eigenvalues.

(c) The characteristic polynomial of **C** is $|\mathbf{C} - \lambda\mathbf{I}| = \lambda^2 - 25$, and we see immediately that the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -5$. The eigenvectors are determined by the equation systems

$$\left.\begin{aligned} x + 4y &= 5x \\ 6x - y &= 5y \end{aligned}\right\} \iff x = y \qquad \text{and} \qquad \left.\begin{aligned} x + 4y &= -5x \\ 6x - y &= -5y \end{aligned}\right\} \iff y = -\frac{3}{2}x$$

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respectively, so the eigenvectors are

$$\mathbf{v}_1 = s\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = t\begin{pmatrix}-2\\3\end{pmatrix}$$

where s and t are arbitrary nonzero numbers.

(d) The characteristic polynomial of $\mathbf{D}$ is

$$|\mathbf{D} - \lambda\mathbf{I}| = \begin{vmatrix}2-\lambda & 0 & 0\\ 0 & 3-\lambda & 0\\ 0 & 0 & 4-\lambda\end{vmatrix} = (2-\lambda)(3-\lambda)(4-\lambda)$$

The eigenvalues are obviously $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$, and the corresponding eigenvectors are

$$\mathbf{v}_1 = s\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{v}_2 = t\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathbf{v}_3 = u\begin{pmatrix}0\\0\\1\end{pmatrix}$$

where s, t, u are arbitrary nonzero numbers. (*Note:* The eigenvalues of a diagonal matrix are always precisely the diagonal elements, and (multiples of) the standard unit vectors will be eigenvectors. But if two or more of the diagonal elements are equal, there will be other eigenvectors as well. An extreme case is the identity matrix $\mathbf{I}_n$: all (nonzero) n-vectors are eigenvectors for $\mathbf{I}_n$.)

(e) The characteristic polynomial of $\mathbf{E}$ is

$$\begin{vmatrix}2-\lambda & 1 & -1\\ 0 & 1-\lambda & 1\\ 2 & 0 & -2-\lambda\end{vmatrix} = -\lambda^3 + \lambda^2 + 2\lambda = -\lambda(\lambda^2 - \lambda - 2)$$

The eigenvalues are the roots of the equation $-\lambda(\lambda^2 - \lambda - 2) = 0$, namely $\lambda_1 = -1$, $\lambda_2 = 0$ and $\lambda_3 = 2$.

The eigenvectors corresponding to $\lambda_1 = -1$ are solutions of

$$\mathbf{E}\mathbf{x} = -\mathbf{x} \iff \begin{cases}2x_1 + x_2 - x_3 = -x_1\\ \phantom{2x_1 +{}} x_2 + x_3 = -x_2\\ 2x_1 \phantom{{}+x_2} - 2x_3 = -x_2\end{cases} \iff \begin{cases}x_1 = \frac{1}{2}x_3\\ x_2 = -\frac{1}{2}x_3\end{cases} \iff \begin{cases}x_2 = -x_1\\ x_3 = 2x_1\end{cases}$$

so they are of the form $\mathbf{v}_1 = s\begin{pmatrix}1\\-1\\2\end{pmatrix}$. Similarly, $\mathbf{v}_2 = t\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\mathbf{v}_3 = u\begin{pmatrix}2\\1\\1\end{pmatrix}$ are the eigenvectors corresponding to $\lambda_2 = 0$ and $\lambda_3 = 2$.

(f) The characteristic polynomial of $\mathbf{F}$ is

$$\begin{vmatrix}1-\lambda & -1 & 0\\ -1 & 2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{vmatrix} = -\lambda^3 + 4\lambda^2 - 3\lambda = -\lambda(\lambda^2 - 4\lambda + 3) = -\lambda(\lambda - 1)(\lambda - 3)$$

The eigenvalues are therefore $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 3$. By the same method as above we find that the corresponding eigenvectors are $\mathbf{v}_1 = s\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\mathbf{v}_2 = t\begin{pmatrix}-1\\0\\1\end{pmatrix}$, and $\mathbf{v}_3 = u\begin{pmatrix}1\\-2\\1\end{pmatrix}$.

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1.5.2 (a) $\mathbf{X}'\mathbf{A}\mathbf{X} = \mathbf{X}'(\mathbf{A}\mathbf{X}) = (x,\ y,\ z)\begin{pmatrix} ax+ay \\ ax+ay \\ bz \end{pmatrix} = (ax^2 + ay^2 + 2axy + bz^2)$ (a 1×1 matrix),

$$\mathbf{A}^2 = \begin{pmatrix} 2a^2 & 2a^2 & 0 \\ 2a^2 & 2a^2 & 0 \\ 0 & 0 & b^2 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 4a^3 & 4a^3 & 0 \\ 4a^3 & 4a^3 & 0 \\ 0 & 0 & b^3 \end{pmatrix}$$

(b) The characteristic polynomial of $\mathbf{A}$ is $p(\lambda) = \begin{vmatrix} a-\lambda & a & 0 \\ a & a-\lambda & 0 \\ 0 & 0 & b-\lambda \end{vmatrix} = (\lambda^2 - 2a\lambda)(b-\lambda)$, so the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$, $\lambda_2 = 2a$, $\lambda_3 = b$.

(c) From (b) we get $p(\lambda) = -\lambda^3 + (2a+b)\lambda^2 - 2ab\lambda$. Using the expressions for $\mathbf{A}^2$ and $\mathbf{A}^3$ that we found in part (a), it is easy to show that $p(\mathbf{A}) = -\mathbf{A}^3 + (2a+b)\mathbf{A}^2 - 2ab\mathbf{A} = \mathbf{0}$.

1.5.4 (a) The formula in Problem 1.9.7(b) yields

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & 1 & 1 & 1 \\ 1 & 4-\lambda & 1 & 1 \\ 1 & 1 & 4-\lambda & 1 \\ 1 & 1 & 1 & 4-\lambda \end{vmatrix} = (3-\lambda)^4\left(1 + \frac{4}{3-\lambda}\right) = (3-\lambda)^3(7-\lambda)$$

Hence, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = \lambda_3 = 3$, $\lambda_4 = 7$.

(b) An eigenvector $\mathbf{x} = (x_1, x_2, x_3, x_4)'$ of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 3$ must satisfy the equation system $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$. The 4 equations in this system are all the same, namely

$$x_1 + x_2 + x_3 + x_4 = 0$$

The system has solutions with $4 - 1 = 3$ degrees of freedom. One simple set of solutions is

$$\mathbf{x}^1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}^2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad \mathbf{x}^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

These three vectors are obviously linearly independent because if

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + c_3\mathbf{x}^3 = \begin{pmatrix} c_1 + c_2 + c_3 \\ -c_1 \\ -c_2 \\ -c_3 \end{pmatrix}$$

is the zero vector, then $c_1 = c_2 = c_3 = 0$.

1.5.5 (c) If λ is an eigenvalue for $\mathbf{C}$ with an associated eigenvector $\mathbf{x}$, then $\mathbf{C}^n\mathbf{x} = \lambda^n\mathbf{x}$ for every natural number n. If $\mathbf{C}^3 = \mathbf{C}^2 + \mathbf{C}$, then $\lambda^3\mathbf{x} = \lambda^2\mathbf{x} + \lambda\mathbf{x}$, so $(\lambda^3 - \lambda^2 - \lambda)\mathbf{x} = \mathbf{0}$. Then $\lambda^3 - \lambda^2 - \lambda = 0$, because $\mathbf{x} \neq \mathbf{0}$. If $\mathbf{C} + \mathbf{I}_n$ did not have an inverse, $|\mathbf{C} + \mathbf{I}_n| = 0$. Then $\lambda = -1$ would be an eigenvalue for $\mathbf{C}$, and so we would have $\lambda^3 - \lambda^2 - \lambda = 0$, which is not true for $\lambda = -1$. Hence -1 is not an eigenvalue for $\mathbf{C}$, and consequently $\mathbf{C} + \mathbf{I}_n$ has an inverse.

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1.6

1.6.1 (a) Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$$

Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. The associated eigenvectors with length 1 are uniquely determined up to sign as

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$$

This yields the orthogonal matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

(It is easy to verify that $\mathbf{P}'\mathbf{P} = \mathbf{I}$, i.e. $\mathbf{P}^{-1} = \mathbf{P}'$.) We therefore have the diagonalization

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

(b) Let $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. The characteristic polynomial of $\mathbf{B}$ is

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\big((1-\lambda)^2 - 1\big) = (2-\lambda)(\lambda^2 - 2\lambda) = -\lambda(\lambda-2)^2$$

(use cofactor expansion of the first determinant along the last row or the last column). The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 2$. It is easily seen that one eigenvector associated with the eigenvalue $\lambda_1 = 0$ is $\mathbf{x}_1 = (1, -1, 0)'$. Eigenvectors $\mathbf{x} = (x_1, x_2, x_3)'$ associated with the eigenvalue 2 are given by $(\mathbf{B} - 2\mathbf{I})\mathbf{x} = \mathbf{0}$, i.e.

$$\begin{aligned} -x_1 + x_2 \quad &= 0 \\ x_1 - x_2 \quad &= 0 \\ 0 &= 0 \end{aligned}$$

This gives $x_1 = x_2$, x_3 arbitrary. One set of linearly independent eigenvectors with length 1 is then

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Fortunately, these three vectors are mutually orthogonal (this is not automatically true for two eigenvectors associated with the same eigenvalue), and so we have a suitable orthogonal matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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It is now easy to verify that $\mathbf{P}^{-1} = \mathbf{P}'$ and

$$\mathbf{P}^{-1}\mathbf{BP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) The characteristic polynomial of $\mathbf{C} = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$ is

$$\begin{vmatrix} 1-\lambda & 3 & 4 \\ 3 & 1-\lambda & 0 \\ 4 & 0 & 1-\lambda \end{vmatrix} = 4\begin{vmatrix} 3 & 1-\lambda \\ 4 & 0 \end{vmatrix} + (1-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)\big((1-\lambda)^2 - 25\big)$$

(cofactor expansion along the last column), and so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 6$, and $\lambda_3 = -4$. An eigenvector $\mathbf{x} = (x, y, z)'$ corresponding to the eigenvalue λ must satisfy

$$\mathbf{Cx} = \lambda\mathbf{x} \iff \begin{aligned} x + 3y + 4z &= \lambda x \\ 3x + y \phantom{{}+4z} &= \lambda y \\ 4x \phantom{{}+3y} + z &= \lambda z \end{aligned}$$

One set of unnormalized eigenvectors is

$$\mathbf{u} = \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix}$$

with lengths $\|\mathbf{u}\| = 5$, $\|\mathbf{v}\| = \|\mathbf{w}\| = 5\sqrt{2}$, and a corresponding orthogonal matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{5}{10}\sqrt{2} & -\frac{5}{10}\sqrt{2} \\ -\frac{4}{5} & \frac{3}{10}\sqrt{2} & \frac{3}{10}\sqrt{2} \\ \frac{3}{5} & \frac{4}{10}\sqrt{2} & \frac{4}{10}\sqrt{2} \end{pmatrix}$$

A straightforward calculation confirms that $\mathbf{P}'\mathbf{CP} = \text{diag}(1, 6, -4) = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

1.6.5 For the given $\mathbf{A}$, we have $\mathbf{A}^2 = 5\mathbf{A} - 5\mathbf{I}$. Therefore $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = (5\mathbf{A} - 5\mathbf{I})\mathbf{A} = 5\mathbf{A}^2 - 5\mathbf{A} = 20\mathbf{A} - 25\mathbf{I}$ and $\mathbf{A}^4 = 20\mathbf{A}^2 - 25\mathbf{A} = 75\mathbf{A} - 100\mathbf{I} = \begin{pmatrix} 50 & 75 \\ 75 & 125 \end{pmatrix}$.

1.7

1.7.5 (a) It is clear that $Q(x_1, x_2) \geq 0$ for all x_1 and x_2 and that $Q(x_1, x_2) = 0$ only if $x_1 = x_2 = 0$, so Q is positive definite.

(b) The symmetric coefficient matrix of Q is $\begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. The leading principal minors are

$$D_1 = 5, \qquad D_2 = \begin{vmatrix} 5 & 0 \\ 0 & 2 \end{vmatrix} = 10, \qquad D_3 = \begin{vmatrix} 5 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 33$$

Since all the leading principal minors are positive, it follows from Theorem 1.7.1 that Q is positive definite. (An alternative way to see this is to write Q as a sum of squares: $Q(x_1, x_2, x_3) = (x_1 + x_3)^2 + (x_2 + x_3)^2 + 4x_1^2 + x_2^2 + 2x_3^2$ is obviously nonnegative and it is zero only if all the square terms are zero. But it is not always easy to see how to rewrite a quadratic form in this fashion.)

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(c) Since $Q(x_1, x_2) = -(x_1 - x_2)^2 \leq 0$ for all x_1 and x_2, Q is negative semidefinite. But Q is not definite, since $Q(x_1, x_2) = 0$ whenever $x_1 = x_2$.

(d) The symmetric coefficient matrix of Q is $\begin{pmatrix} -3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & -8 \end{pmatrix}$, and the leading principal minors are $D_1 = -3 < 0$, $D_2 = 2 > 0$, and $D_3 = -4 < 0$. By Theorem 1.7.1, Q is negative definite.

1.7.7 (a) The symmetric coefficient matrix of Q is

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 3 & -(5+c)/2 \\ -(5+c)/2 & 2c \end{pmatrix}$$

Since $a_{11} > 0$, the form can never be negative semidefinite. It is

(i) positive definite if $a_{11} > 0$ and $|\mathbf{A}| > 0$,

(ii) positive semidefinite if $a_{11} \geq 0$, $a_{22} \geq 0$. and $|\mathbf{A}| \geq 0$,

(iii) indefinite if $|\mathbf{A}| < 0$.

The determinant of $\mathbf{A}$ is

$$|\mathbf{A}| = 6c - \tfrac{1}{4}(5+c)^2 = -\tfrac{1}{4}(c^2 - 14c + 25) = -\tfrac{1}{4}(c - c_1)(c - c_2)$$

where $c_1 = 7 - 2\sqrt{6} \approx 2.101$ and $c_2 = 7 + 2\sqrt{6} \approx 11.899$ are the roots of the quadratic equation $c^2 - 14c + 25 = 0$. It follows that

(1) Q is positive definite if $c_1 < c < c_2$, i.e. if c lies in the open interval (c_1, c_2);

(2) Q is positive semidefinite if $c_1 \leq c \leq c_2$, i.e. if c lies in the closed interval $[c_1, c_2]$;

(3) Q is indefinite if $c < c_1$ or $c > c_2$, i.e. if c lies outside the closed interval $[c_1, c_2]$.

1.7.10 (b) If $\mathbf{x}$ satisfies the Lagrange conditions, then $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'(\lambda\mathbf{x}) = \lambda\mathbf{x}'\mathbf{x} = \lambda\|\mathbf{x}\|^2 = \lambda$, because the constraint is simply $\|\mathbf{x}\|^2 = 1$. Hence, the maximum and minimum values of $Q(\mathbf{x})$ are simply the largest and smallest eigenvalue of Q, which are $\lambda_1 = 9$ and $\lambda_2 = -5$. The corresponding maximum and minimum points (eigenvectors) are $\pm\frac{1}{2}\sqrt{2}\,(1, 1)$ and $\pm\frac{1}{2}\sqrt{2}\,(1, -1)$, respectively.

1.8

1.8.3 Negative definite subject to the constraint, by Theorem 1.8.1: $\begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 4 & -2 & 1 \\ 1 & 4 & -5 & 1 & 2 \\ 1 & -2 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 & -2 \end{vmatrix} = -180 <$ 0.

1.8.4 Positive definite subject to the constraint, by Theorem 1.8.1: $\begin{vmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 & 1 \end{vmatrix} = 25 > 0.$

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1.9

1.9.3 The obvious partitioning to use in (a) is with $\mathbf{A}_{11}$ as 2×2 matrix in the upper left corner of $\mathbf{A}$, and in (b) it is natural to let $\mathbf{A}_{11}$ be the upper left 1×1 matrix. In both cases the partitioned matrix will be of the form $\begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}$, i.e. $\mathbf{A}_{12}$ and $\mathbf{A}_{21}$ will both be zero matrices. If we use formula (1.9.4) we get $\boldsymbol{\Delta} = \mathbf{A}_{22}$, whereas formula (1.9.5) gives $\tilde{\boldsymbol{\Delta}} = \mathbf{A}_{11}$. In either case we find that $\begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{pmatrix}$, and the answers to (a) and (b) are as given in the book.

The matrix in (c) can be partitioned as $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{v} \\ \mathbf{v}' & \mathbf{I}_1 \end{pmatrix}$, where $\mathbf{I}_4$ and $\mathbf{I}_1$ are the identity matrices of orders 4 and 1, respectively, and $\mathbf{v}' = (1, 1, 1, 0)$ is the transpose of the 4×1 matrix (column vector) $\mathbf{v}$. If we use formula (1.9.4) in the book, we get $\boldsymbol{\Delta} = \mathbf{I}_1 - \mathbf{v}'\mathbf{v} = (-2) = -2\mathbf{I}_1$.

Then $\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\boldsymbol{\Delta}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} = \mathbf{I}_4 - \frac{1}{2}\mathbf{v}\mathbf{v}' = \mathbf{I}_4 - \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$,

and $-\boldsymbol{\Delta}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} = \frac{1}{2}\mathbf{I}_1\mathbf{v}'\mathbf{I}_4 = \frac{1}{2}\mathbf{v}' = \frac{1}{2}(1, 1, 1, 0)$. Further, $-\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\boldsymbol{\Delta}^{-1} = \frac{1}{2}\mathbf{v}$, so we finally get

$$\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{I}_4 - \frac{1}{2}\mathbf{v}\mathbf{v}' & \frac{1}{2}\mathbf{v} \\ \frac{1}{2}\mathbf{v}' & -\frac{1}{2}\mathbf{I}_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & -1 \end{pmatrix}.$$

An alternative partitioning is $\mathbf{A} = \begin{pmatrix} \mathbf{I}_3 & \mathbf{W} \\ \mathbf{W}' & \mathbf{I}_2 \end{pmatrix}$, where $\mathbf{W}' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Then $\boldsymbol{\Delta} = \mathbf{I}_2 - \mathbf{W}\mathbf{W}' = \mathbf{I}_2 - \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, so $\boldsymbol{\Delta}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. A bit of calculation shows that $-\boldsymbol{\Delta}^{-1}\mathbf{W}' = \frac{1}{2}\mathbf{W}$ and $\mathbf{W}\mathbf{W}' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, so $\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{I}_3 - \frac{1}{2}\mathbf{W}\mathbf{W}' & \frac{1}{2}\mathbf{W} \\ \frac{1}{2}\mathbf{W}' & \boldsymbol{\Delta}^{-1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\mathbf{I}_3 - \mathbf{W}\mathbf{W}' & \mathbf{W} \\ \mathbf{W}' & 2\boldsymbol{\Delta}^{-1} \end{pmatrix}$, as before.

(Note that because $\mathbf{A}$ is symmetric, $\mathbf{A}^{-1}$ must also be symmetric. Thus the upper right submatrix of $\mathbf{A}^{-1}$ must be the transpose of the lower left submatrix, which helps us save a little work.)

1.9.4 The matrix $\mathbf{B} = \begin{pmatrix} 1 & -x_1 & \dots & -x_n \\ x_1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & a_{n1} & \dots & a_{nn} \end{pmatrix}$ can be partitioned as $\mathbf{B} = \begin{pmatrix} \mathbf{I}_1 & -\mathbf{X}' \\ \mathbf{X} & \mathbf{A} \end{pmatrix}$, where $\mathbf{A}$ and $\mathbf{X}$ are as given in the problem. We evaluate the determinant of $\mathbf{B}$ by each of the formulas (6) and (7): By formula (6),

$$|\mathbf{B}| = |\mathbf{I}_1| \cdot |\mathbf{A} - \mathbf{X}\mathbf{I}_1^{-1}(-\mathbf{X}')| = |\mathbf{A} + \mathbf{X}\mathbf{X}'|$$

and by formula (7),

$$|\mathbf{B}| = |\mathbf{A}| \cdot |\mathbf{I}_1 - (-\mathbf{X}')\mathbf{A}^{-1}\mathbf{X}| = |\mathbf{A}| \cdot |\mathbf{I}_1 + \mathbf{X}'\mathbf{A}^{-1}\mathbf{X}|$$

where the last factor is the determinant of a 1×1 matrix and therefore equal to the single element $1 + \mathbf{X}'\mathbf{A}^{-1}\mathbf{X}$ of that matrix.

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1.9.6 (a) Let $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}$, where $\mathbf{A}_{11}$ is a $k \times k$ matrix and $\mathbf{A}_{22}$ is $(n-k) \times (n-k)$. We will consider two ways to demonstrate that $|\mathbf{A}| = |\mathbf{A}_{11}|\,|\mathbf{A}_{22}|$.

(I) By definition, the determinant of the $n \times n$ matrix $\mathbf{A}$ is a sum Σ_A of $n!$ terms. Each of these terms is the product of a sign factor and n elements from $\mathbf{A}$, chosen so that no two elements are taken from the same row of $\mathbf{A}$ or from the same column. The sign factor ± 1 is determined by the positions of the elements selected. (See EMEA or almost any book on linear algebra for details.)

A term in this sum will automatically be zero unless the factors from the first k columns are taken from $\mathbf{A}_{11}$ and the last $n-k$ factors from $\mathbf{A}_{22}$. If the factors are selected in this way, the term in question will be a product of one term in the sum Σ_1 making up $|\mathbf{A}_{11}|$ and one from the sum Σ_2 that makes up $|\mathbf{A}_{22}|$. (The sign factors will match.) All such pairs of terms will occur exactly once and so $\Sigma_A = \Sigma_1 \Sigma_2$, that is, $|\mathbf{A}| = |\mathbf{A}_{11}|\,|\mathbf{A}_{22}|$.

(II) Suppose that $\mathbf{A}$ is upper triangular, i.e. all elements below the main diagonal are 0. Then $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are also upper triangular. We know that the determinant of a triangular matrix equals the product of the elements on the main diagonal. (Just think of cofactor expansion along the first column, then the second column, etc.) In this case it is clear that $|\mathbf{A}| = |\mathbf{A}_{11}|\,|\mathbf{A}_{22}|$.

Of course, in the general case $\mathbf{A}$ need not be upper triangular at all, but we can make it so by means of elementary row operations, more specifically the operations of (i) adding a multiple of one row to another row and (ii) interchanging two rows. Operation (i) does not affect the value of the determinant, while operation (ii) multiplies the value by -1. We perform such operations on the first k rows of $\mathbf{A}$ in such a way that $\mathbf{A}_{11}$ becomes upper triangular, and then operate on the last $n-k$ rows of $\mathbf{A}$ such that $\mathbf{A}_{22}$ becomes upper triangular. The number σ of sign changes that $|\mathbf{A}|$ undergoes is then the sum of the numbers σ_1 and σ_2 of sign changes inflicted on $|\mathbf{A}_{11}|$ and $|\mathbf{A}_{22}|$, respectively. By the formula we showed for the upper triangular case, $(-1)^{\sigma}|\mathbf{A}| = (-1)^{\sigma_1}|\mathbf{A}_{11}| \cdot (-1)^{\sigma_2}|\mathbf{A}_{22}|$, and since $\sigma_1 + \sigma_2 = \sigma$, we get $|\mathbf{A}| = |\mathbf{A}_{11}|\,|\mathbf{A}_{22}|$ in the general case too.

To show the formula $\begin{vmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{vmatrix} = |\mathbf{A}_{11}|\,|\mathbf{A}_{22}|$, simply look at the determinant $\begin{vmatrix} \mathbf{A}'_{11} & \mathbf{A}'_{21} \\ \mathbf{0} & \mathbf{A}'_{22} \end{vmatrix}$ of the transposed matrix.

(b) The equality follows by direct multiplication. By the result in (a), the first factor on the left has determinant $|\mathbf{I}_k|\,|\mathbf{I}_{n-k}| = 1$, and so, by (a) again,

$$\begin{vmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{vmatrix} = \begin{vmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{vmatrix} = |\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}|\,|\mathbf{A}_{22}|$$

1.9.7 (a) With $\mathbf{D} = \begin{pmatrix} \mathbf{I}_n & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_m \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} \mathbf{I}_n & -\mathbf{A} \\ \mathbf{B} & \mathbf{I}_m \end{pmatrix}$ we get

$$\mathbf{DE} = \begin{pmatrix} \mathbf{I}_n + \mathbf{AB} & \mathbf{0} \\ \mathbf{B} & \mathbf{I}_m \end{pmatrix} \quad \text{and} \quad \mathbf{ED} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{B} & \mathbf{I}_m + \mathbf{BA} \end{pmatrix}$$

Cofactor expansion of the determinant of $\mathbf{DE}$ along each of the last m columns shows that $|\mathbf{DE}| = |\mathbf{I}_n + \mathbf{AB}|$. Similarly, cofactor expansion along each of the first n rows shows that $|\mathbf{ED}| = |\mathbf{I}_m + \mathbf{BA}|$.

Alternatively, we could use formula (7) with $\mathbf{A}_{22} = \mathbf{I}_m$ to evaluate $|\mathbf{DE}|$ and formula (6) with $\mathbf{A}_{11} = \mathbf{I}_n$ to evaluate $|\mathbf{ED}|$.

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(b) With $\mathbf{A}$ and $\mathbf{B}$ as in the hint, $\mathbf{AB}$ is an $n \times n$ matrix with every column equal to $\mathbf{A}$. Therefore

$$\mathbf{F} = \mathbf{I}_n + \mathbf{AB} = \begin{pmatrix} \dfrac{a_1}{a_1 - 1} & \dfrac{1}{a_1 - 1} & \dots & \dfrac{1}{a_1 - 1} \\ \dfrac{1}{a_2 - 1} & \dfrac{a_2}{a_2 - 1} & \dots & \dfrac{1}{a_2 - 1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{a_n - 1} & \dfrac{1}{a_n - 1} & \dots & \dfrac{a_n}{a_n - 1} \end{pmatrix}$$

and

$$\mathbf{G} = \begin{pmatrix} a_1 & 1 & \dots & 1 \\ 1 & a_2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & a_n \end{pmatrix} = \begin{pmatrix} a_1 - 1 & 0 & \dots & 0 \\ 0 & a_2 - 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n - 1 \end{pmatrix} (\mathbf{I}_n + \mathbf{AB})$$

From the result in (a) it follows that

$$\begin{aligned} |\mathbf{G}| &= (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)\, |\mathbf{I}_n + \mathbf{AB}| = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)\, |\mathbf{I}_1 + \mathbf{BA}| \\ &= (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)\Big(1 + \sum_{i=1}^{n} \frac{1}{a_i - 1}\Big) \end{aligned}$$

Chapter 2 Multivariable Calculus

2.1

2.1.3 (a) The unit vector in the direction given by $\mathbf{v} = (1, 1)$ is $\mathbf{a} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{2}}(1, 1)$. By formula (2.1.8) the directional derivative of f in this direction at $(2, 1)$ is

$$f'_{\mathbf{a}}(2, 1) = \nabla f(2, 1) \cdot \mathbf{a} = \frac{1}{\sqrt{2}} \nabla f(2, 1) \cdot (1, 1) = \frac{1}{\sqrt{2}}(2, 1) \cdot (1, 1) = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Note: It is pure coincidence that the gradient of f at (2,1) equals (2,1).

(b) The gradient of g is $\nabla g(x, y, z) = ((1 + xy)e^{xy} - y, x^2 e^{xy} - x, -2z)$, and the unit vector in the direction given by $(1, 1, 1)$ is $\mathbf{b} = \frac{1}{\sqrt{3}}(1, 1, 1)$. Formula (2.8.1) gives

$$g'_{\mathbf{b}}(0, 1, 1) = \nabla g(0, 1, 1) \cdot \mathbf{b} = \frac{1}{\sqrt{3}}(0, 0, -2) \cdot (1, 1, 1) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

2.1.5 (a) The vector from $(3, 2, 1)$ to $(-1, 1, 2)$ is $(-1, 1, 2) - (3, 2, 1) = (-4, -1, 1)$, and the unit vector in this direction is $\mathbf{a} = \frac{1}{\sqrt{18}}(-4, -1, 1)$. The gradient of f is

$$\nabla f(x, y, z) = \Big(y \ln(x^2 + y^2 + z^2) + \frac{2x^2 y}{x^2 + y^2 + z^2},\ x \ln(x^2 + y^2 + z^2) + \frac{2xy^2}{x^2 + y^2 + z^2},\ \frac{2xyz}{x^2 + y^2 + z^2}\Big)$$

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By formula (2.1.8) the directional derivative of f at $(1, 1, 1)$ in the direction $\mathbf{a}$ is

$$f'_{\mathbf{a}}(1,1,1) = (1/\sqrt{18})\,\nabla f(1,1,1)\cdot(-4,-1,1)$$
$$= (1/\sqrt{18})\,(\ln 3 + 2/3, \ln 3 + 2/3, 2/3)\cdot(-4,-1,1) = -(5\ln 3 + 8/3)/\sqrt{18}$$

(b) At $(1, 1, 1)$ the direction of fastest growth for f is given by $\nabla f(1,1,1) = (\ln 3+2/3, \ln 3+2/3, 2/3)$.

2.1.6 The unit vector in the direction of maximal increase of f at $(0, 0)$ is $\mathbf{a} = \frac{1}{\sqrt{10}}(1,3)$ and therefore $\nabla f(0,0) = t\mathbf{a}$ for a number $t \geq 0$. We also know that $f'_{\mathbf{a}}(0,0) = 4$. On the other hand, by (2.1.8), $f'_{\mathbf{a}}(0,0) = \nabla f(0,0)\cdot\mathbf{a} = t\mathbf{a}\cdot\mathbf{a} = t\|\mathbf{a}\|^2 = t$. Hence, $t = 4$ and

$$\nabla f(0,0) = 4\mathbf{a} = \frac{4}{\sqrt{10}}(1,3) = \frac{4\sqrt{10}}{10}(1,3) = \frac{2\sqrt{10}}{5}(1,3)$$

2.1.9 (a) We know that $y' = -F'_1(x,y)/F'_2(x,y)$, and by the chain rule for functions of several variables,

$$y'' = \frac{d}{dx}(y') = -\frac{\partial}{\partial x}\left(\frac{F'_1(x,y)}{F'_2(x,y)}\right)\frac{dx}{dx} - \frac{\partial}{\partial y}\left(\frac{F'_1(x,y)}{F'_2(x,y)}\right)\frac{dy}{dx}$$
$$= -\frac{F''_{11}F'_2 - F'_1F''_{21}}{(F'_2)^2}\cdot 1 - \frac{F''_{12}F'_2 - F'_1F''_{22}}{(F'_2)^2}\left(-\frac{F'_1}{F'_2}\right)$$
$$= \frac{-(F''_{11}F'_2 - F'_1F''_{21})F'_2 + (F''_{12}F'_2 - F'_1F''_{22})F'_1}{(F'_2)^3} = \frac{-F''_{11}(F'_2)^2 + 2F''_{12}F'_1F'_2 - F''_{22}(F'_1)^2}{(F'_2)^3}$$

(Remember that $F''_{21} = F''_{12}$.) Expanding the determinant in the problem yields precisely the numerator in the last fraction.

2.2

2.2.6 (a) If $\mathbf{x}$ and $\mathbf{y}$ are points in S and $\lambda \in [0,1]$, then $\|\mathbf{x}\| \leq r$, $\|\mathbf{y}\| \leq r$, and by the triangle inequality, $\|\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\| \leq \lambda\|\mathbf{x}\| + (1-\lambda)\|\mathbf{y}\| \leq \lambda r + (1-\lambda)r = r$. Hence, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ belongs to S. It follows that S is convex.

(b) S_1 is the interior of the ball S, i.e. what we get when we remove the spherical "shell" $\{\mathbf{x} : \|\mathbf{x}\| = r\}$ from S. The triangle inequality shows that S_1 is convex. The set S_2 is the spherical shell we mentioned, while S_3 consists of the shell and the part of $\mathbb{R}^n$ that lies outside S. Neither S_2 nor S_3 is convex.

2.2.7 (a) Let us call a set S of numbers *midpoint convex* if $\frac{1}{2}(x_1 + x_2)$ whenever x_1 and x_2 belong to S. The set $S = \mathbb{Q}$ of rational numbers is midpoint convex, but it is not convex, for between any two rational numbers r_1 and r_2 there are always irrational numbers. For example, let $t = r_1 + (r_2 - r_1)/\sqrt{2}$. Then t lies between r_1 and r_2. If t were rational, then $\sqrt{2} = (r_2 - r_1)/(t - r_1)$ would also be rational, but we know that $\sqrt{2}$ is irrational.

(b) Suppose S is midpoint convex and closed. Let x_1 and x_2 be points in S, with $x_1 < x_2$, and let λ in $(0, 1)$. We shall prove that the point $z = \lambda x_1 + (1-\lambda)x_2$ belongs to S. Since S is midpoint convex, it must contain $y_1 = \frac{1}{2}(x_1 + x_2)$, and then it contains $y_{21} = \frac{1}{2}(x_1 + y_1) = \frac{3}{4}x_1 + \frac{1}{4}x_2$ and $y_{22} = \frac{1}{2}(y_1 + x_2) = \frac{1}{4}x_1 + \frac{3}{4}x_2$. We can continue in this fashion, constructing new midpoints between the points that have already been constructed, and we find that S must contain all points of the form

$$\frac{k}{2^n}x_1 + \frac{2^n - k}{2^n}x_2 = \frac{k}{2^n}x_1 + \left(1 - \frac{k}{2^n}\right)x_2, \qquad n = 1, 2, 3, \ldots, \qquad k = 0, 1, \ldots, 2^n$$

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Thus, for each n we find $2^n + 1$ evenly spaced points in the interval $[x_1, x_2]$ that all belong to S. The distance between two neighbouring points in this collection is $(x_2 - x_1)/2^n$. Now let r be any positive number, and consider the open r-ball (open interval) $B = B(z; r) = (z - r, z + r)$ around z. Let n be so large that $(x_2 - x_1)/2^n < 2r$. Then B must contain at least one of the points $(k/2^n)x_1 + (1 - k/2^n)x_2$ constructed above, so B contains at least one point from S. It follows that z does indeed belong to $\mathrm{cl}(S) = S$.

2.2.8 Let S be a convex subset of $\mathbb{R}$ containing more than one point, and let $a = \inf S$, $b = \sup S$ (where a and b may be finite or infinite). Then $a < b$. In order to prove that S is an interval, it suffices to show that S contains the open interval (a, b). Let x be a point in (a, b). Since $x < b = \sup S$, there exists a β in S with $x < \beta$. Similarly, there is an α in S with $\alpha < x$. Then x is a convex combination of α and β, and since S is convex, x belongs to S.

2.3

2.3.5 (a) $z''_{11}(x, y) = -e^x - e^{x+y}$, $z''_{12}(x, y) = -e^{x+y}$, and $z''_{22}(x, y) = -e^{x+y}$, so $z''_{11}(x, y) < 0$ and $z''_{11}z''_{22} - (z''_{12})^2 = e^{2x+y} > 0$. By Theorem 2.3.1, z is a strictly concave function of x and y.

(b) z is strictly convex, because $z''_{11} = e^{x+y} + e^{x-y} > 0$ and $z''_{11}z''_{22} - (z''_{12})^2 = 4e^{2x} > 0$.

(c) $w = u^2$, where $u = x + 2y + 3z$. So w is a convex function of an affine function, hence convex according to (2.3.8). It is not strictly convex, however, because it is constant on every plane of the form $x + 2y + 3z = c$, and therefore constant along each line joining two points in such a plane.

2.3.6 (b) Let $\lambda_2 > \lambda_1 > 0$ and define $\mu = \lambda_1/\lambda_2$. Then $\mu \in (0, 1)$ and by the concavity of f we have $\mu f(\lambda_2\mathbf{x}) + (1 - \mu)f(\mathbf{0}) \leq f(\mu\lambda_2\mathbf{x} + (1 - \mu)\mathbf{0})$, i.e. $(\lambda_1/\lambda_2)f(\lambda_2\mathbf{x}) \leq f(\lambda_1\mathbf{x})$, so $f(\lambda_2\mathbf{x})/\lambda_2 \leq f(\lambda_1\mathbf{x})/\lambda_1$. It also follows that if f is *strictly* concave, then $f(\lambda\mathbf{x})/\lambda$ is strictly decreasing as a function of λ.

(c) Take any $\mathbf{x} \neq \mathbf{0}$ in the domain of f. Then $\mathbf{x} \neq 2\mathbf{x}$ and $f(\frac{1}{2}\mathbf{x} + \frac{1}{2}2\mathbf{x}) = f(\frac{3}{2}\mathbf{x}) = \frac{3}{2}f(\mathbf{x}) = \frac{1}{2}f(\mathbf{x}) + f(\mathbf{x}) = \frac{1}{2}f(\mathbf{x}) + \frac{1}{2}f(2\mathbf{x})$. Therefore f cannot be strictly concave.

2.3.8 The challenge here is mainly in getting the derivatives right, but with care and patience you will find that

$$f''_{11}(v_1, v_2) = -Pv_2^2, \quad f''_{12}(v_1, v_2) = Pv_1v_2, \quad f''_{22}(v_1, v_2) = -Pv_1^2$$

where

$$P = (\rho + 1)\delta_1\delta_2 A(v_1v_2)^{-\rho-2}\big(\delta_1 v_1^{-\rho} + \delta_2 v_2^{-\rho}\big)^{-(1/\rho)-2}$$

These formulas show that for all v_1 and v_2, the "direct" second derivatives f''_{11} and f''_{22} have the same sign as $-(\rho + 1)$. Also, $f''_{11}f''_{22} - (f''_{12})^2 = 0$ everywhere. It follows from Theorem 2.3.1 that f is convex if $\rho \leq -1$, and concave if $\rho \geq -1$. If $\rho = -1$ then $f(v_1, v_2) = A(\delta_1 v_1 + \delta_2 v_2)$ is a linear function, which indeed is both convex and concave.

The equation $f''_{11}f''_{22} - (f''_{12})^2 = 0$ is a consequence of the fact that f is homogeneous of degree 1, see e.g. Section 12.6 on homogeneous functions in EMEA. Since f is homogeneous of degree 1, it is linear along each ray from the origin and therefore it cannot be strictly convex or strictly concave for any value of ρ.

2.3.9 (a) This is mainly an exercise in manipulating determinants. If you feel that the calculations below look frightening, try to write them out in full for the case $k = 3$ (or $k = 2$). Note that $z''_{ij} = a_ia_jz/x_ix_j$

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for $i \neq j$, and $z''_{ii} = a_i(a_i - 1)z/x_i^2$. Rule (1.1.20) tells us that a common factor in any column (or row) in a determinant can be "moved outside". Therefore,

$$D_k = \begin{vmatrix} z''_{11} & z''_{12} & \dots & z''_{1k} \\ z''_{21} & z''_{22} & \dots & z''_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ z''_{k1} & z''_{k2} & \dots & z''_{kk} \end{vmatrix} = \begin{vmatrix} \dfrac{a_1(a_1-1)}{x_1^2}z & \dfrac{a_1a_2}{x_1x_2}z & \dots & \dfrac{a_1a_k}{x_1x_k}z \\ \dfrac{a_2a_1}{x_2x_1}z & \dfrac{a_2(a_2-1)}{x_2^2}z & \dots & \dfrac{a_2a_k}{x_2x_k}z \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{a_ka_1}{x_kx_1}z & \dfrac{a_ka_2}{x_kx_2}z & \dots & \dfrac{a_k(a_k-1)}{x_k^2}z \end{vmatrix}$$

$$\overset{(1)}{=} \frac{a_1a_2\dots a_k}{x_1x_2\dots x_k}z^k \begin{vmatrix} \dfrac{a_1-1}{x_1} & \dfrac{a_1}{x_1} & \dots & \dfrac{a_1}{x_1} \\ \dfrac{a_2}{x_2} & \dfrac{a_2-1}{x_2} & \dots & \dfrac{a_2}{x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{a_k}{x_k} & \dfrac{a_k}{x_k} & \dots & \dfrac{a_k-1}{x_k} \end{vmatrix} \overset{(2)}{=} \frac{a_1a_2\dots a_k}{(x_1x_2\dots x_k)^2}z^k \begin{vmatrix} a_1-1 & a_1 & \dots & a_1 \\ a_2 & a_2-1 & \dots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_k & \dots & a_k-1 \end{vmatrix}$$

where equality (1) holds because a_jz/x_j is a common factor in column j for each j and equality (2) holds because $1/x_i$ is a common factor in row i for each i.

(b) More determinant calculations. Let $s_k = \sum_{i=1}^k a_i = a_1 + \dots + a_k$. We use the expression for D_k that we found in part (a), and add rows 2, 3, ..., k to the first row. Then each entry in the first row becomes equal to $s_k - 1$. Afterwards we take the common factor $s_k - 1$ in row 1 and move it outside.

$$D_k = \frac{a_1a_2\dots a_k}{(x_1x_2\dots x_k)^2}z^k \begin{vmatrix} s_k-1 & s_k-1 & \dots & s_k-1 \\ a_2 & a_2-1 & \dots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_k & \dots & a_k-1 \end{vmatrix}$$

$$= (s_k-1)\frac{a_1a_2\dots a_k}{(x_1x_2\dots x_k)^2}z^k \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_2 & a_2-1 & \dots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_k & \dots & a_k-1 \end{vmatrix}$$

Now subtract column 1 from all the other columns. Rule (1.1.22) says that this does not change the value of the determinant, so

$$D_k = (s_k-1)\frac{a_1a_2\dots a_k}{(x_1x_2\dots x_k)^2}z^k \begin{vmatrix} 1 & 0 & \dots & 0 \\ a_2 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_k & 0 & \dots & -1 \end{vmatrix} = (-1)^{k-1}(s_k-1)\frac{a_1a_2\dots a_k}{(x_1x_2\dots x_k)^2}z^k$$

(c) By assumption, $a_i > 0$ for all i, so if $\sum_{i=1}^n a_i < 1$, then $s_k = \sum_{i=1}^k a_i < 1$ for all k. Therefore D_k has the same sign as $(-1)^k$. It follows from Theorem 2.3.2(b) that f is strictly concave.

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2.4

2.4.3 We shall show that Jensen's inequality (2.4.2) holds for all natural numbers m, not just for $m = 3$. Let f be a function defined on a convex set S in $\mathbb{R}^n$ and for each natural number m let $A(m)$ be the following statement: "The inequality

$$f(\lambda_1 \mathbf{x}_1 + \cdots + \lambda_m \mathbf{x}_m) \geq \lambda_1 f(\mathbf{x}_1) + \cdots + \lambda_m f(\mathbf{x}_m)$$

holds for all $\mathbf{x}_1, \ldots, \mathbf{x}_m$ in S and all $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$."

We shall prove that $A(m)$ is true for every natural number m. It is obvious that $A(1)$ is true, since it just says that $f(\mathbf{x}) = f(\mathbf{x})$, and $A(2)$ is also true, since f is concave. Now suppose that $A(k)$ is true, where k is some natural number greater than 1. We shall prove that $A(k+1)$ is also true.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}$ be points in S and let $\lambda_1, \ldots, \lambda_{k+1}$ be nonnegative numbers with sum 1. We can assume that $\lambda_{k+1} > 0$, for otherwise we are really in the case $m = k$. Then $\mu = \lambda_k + \lambda_{k+1} > 0$ and we can define $\mathbf{y} = (1/\mu)(\lambda_k \mathbf{x}_k + \lambda_{k+1}\mathbf{x}_{k+1})$. The point $\mathbf{y}$ is a convex combination of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$, and so $\mathbf{y} \in S$. By $A(k)$ we have

$$\begin{aligned} f(\lambda_1 \mathbf{x}_1 + \cdots + \lambda_{k+1}\mathbf{x}_{k+1}) &= f(\lambda_1 \mathbf{x}_1 + \cdots + \lambda_{k-1}\mathbf{x}_{k-1} + \mu \mathbf{y}) \\ &\geq \lambda_1 f(\mathbf{x}_1) + \cdots + \lambda_{k-1} f(\mathbf{x}_{k-1}) + \mu f(\mathbf{y}) \end{aligned} \tag{$*$}$$

Moreover, since f is concave, and $(\lambda_k/\mu) + (\lambda_{k+1}/\mu) = 1$,

$$\mu f(\mathbf{y}) = \mu f\left(\frac{\lambda_k}{\mu}\mathbf{x}_k + \frac{\lambda_{k+1}}{\mu}\mathbf{x}_{k+1}\right) \geq \mu\left(\frac{\lambda_k}{\mu} f(\mathbf{x}_k) + \frac{\lambda_{k+1}}{\mu} f(\mathbf{x}_{k+1})\right) = \lambda_k f(\mathbf{x}_k) + \lambda_{k+1} f(\mathbf{x}_{k+1})$$

and this inequality together with $(*)$ yields

$$f(\lambda_1 \mathbf{x}_1 + \cdots + \lambda_{k+1}\mathbf{x}_{k+1}) \geq \lambda_1 f(\mathbf{x}_1) + \cdots + \lambda_{k-1} f(\mathbf{x}_{k-1}) + \lambda_k f(\mathbf{x}_k) + \lambda_{k+1} f(\mathbf{x}_{k+1})$$

which shows that $A(k+1)$ is true. To sum up, we have shown that: (i) $A(2)$ is true, (ii) $A(k) \Rightarrow A(k+1)$ for every natural number $k \geq 2$. It then follows by induction that $A(m)$ is true for all $m \geq 2$. Note that both (i) and (ii) are necessary for this conclusion.

2.4.5 Let $\mathbf{x}, \mathbf{y}$ belong to S and let $\lambda \in [0, 1]$. Then $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ belongs to S. To show that f is concave, it is sufficient to show that $\lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \leq f(\mathbf{z})$. By assumption, f has a supergradient $\mathbf{p}$ at $\mathbf{z}$, and therefore

$$f(\mathbf{x}) - f(\mathbf{z}) \leq \mathbf{p} \cdot (\mathbf{x} - \mathbf{z}) \qquad \text{and} \qquad f(\mathbf{y}) - f(\mathbf{z}) \leq \mathbf{p} \cdot (\mathbf{y} - \mathbf{z})$$

Since both λ and $1 - \lambda$ are nonnegative, it follows that

$$\begin{aligned} \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) - f(\mathbf{z}) &= \lambda[f(\mathbf{x}) - f(\mathbf{z})] + (1-\lambda)[f(\mathbf{y}) - f(\mathbf{z})] \\ &\leq \mathbf{p} \cdot [\lambda(\mathbf{x} - \mathbf{z}) + (1-\lambda)(\mathbf{y} - \mathbf{z})] = \mathbf{p} \cdot \mathbf{0} = 0 \end{aligned}$$

by the definition of $\mathbf{z}$.

2.4.6 Theorem 2.4.1(c) tells us that $f(x, y) = x^4 + y^4$ is a strictly convex function of (x, y) if and only if $f(x, y) - f(x_0, y_0) > \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$ whenever $(x, y) \neq (x_0, y_0)$. Since $\nabla f(x_0, y_0) = (4x_0^3, 4y_0^3)$, we have

$$f(x, y) - f(x_0, y_0) - \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = p(x, x_0) + p(y, y_0)$$

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where $p(t, t_0) = t^4 - t_0^4 - 4t_0^3(t - t_0)$ for all t and t_0. Now, $p(t, t_0) = (t - t_0)(t^3 + t^2 t_0 + t t_0^2 - 3t_0^3)$. The second factor here is 0 if $t = t_0$, so it is divisible by $t - t_0$, and polynomial division yields

$$p(t, t_0) = (t - t_0)(t - t_0)(t^2 + 2tt_0 + 3t_0^2) = (t - t_0)^2[(t + t_0)^2 + 2t_0^2]$$

The expression in square brackets is strictly positive unless $t = t_0 = 0$, so it follows that $p(t, t_0) > 0$ whenever $t \neq t_0$. Hence, $p(x, x_0) + p(y, y_0) > 0$ unless both $x = x_0$ and $y = y_0$.

2.5

2.5.2 (a) f is linear, so it is concave and therefore also quasiconcave.

(c) The set of points for which $f(x, y) \geq -1$ is $P_{-1} = \{(x, y) : y \leq x^{-2/3}\}$, which is not a convex set (see Fig. M2.5.2(c), where P_{-1} is the unshaded part of the plane), so f is not quasiconcave. (It is quasiconvex in the first quadrant, though.)

(d) The polynomial $x^3 + x^2 + 1$ is increasing in the interval $(-\infty, -2/3]$, and decreasing in $[-2/3, 0]$. So f is increasing in $(-\infty, -2/3]$ and decreasing in $[-2/3, \infty)$. (See Fig. M2.5.2(d).) Then the upper level sets must be intervals (or empty), and it follows that f is quasiconcave. Alternatively, we could use the result in the note below.

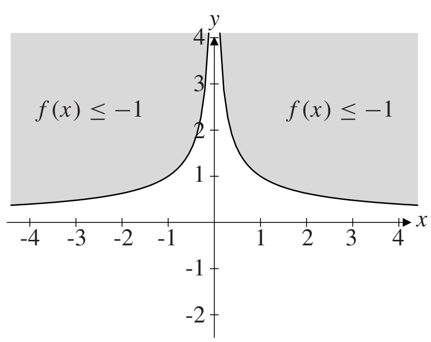

Figure M2.5.2(c) Neither P^a nor P_a is convex for $a = -1$.

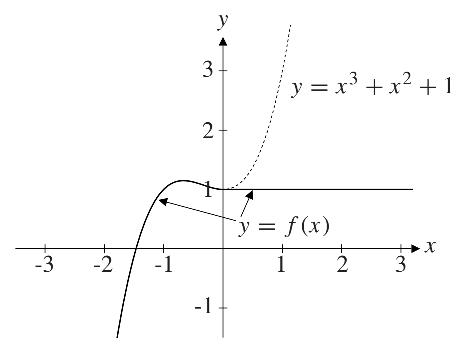

Figure M2.5.2(d) The graph of f.

A note on quasiconcave functions of one variable

It is shown in Example 2.5.2 in the book that a function of one variable that is increasing or decreasing on a whole interval is both quasiconcave and quasiconvex on that interval. Now suppose f is a function defined on an interval (a, b) and that there is a point c in $(a, b$ such that f is increasing on $(a, c]$ and decreasing on $[c, b)$. Then f is quasiconcave on the interval (a, b). This follows because the upper level sets must be intervals. Alternatively we can use Theorem 2.5.1(a) and note that if x and y are points in (a, b), then $f(z) \geq \min\{f(x), f(y)\}$ for all z between x and y. We just have to look at each of the three possibilities $x < y \leq c$, $x \leq c < y$, and $c < x < y$, and consider the behaviour of f over the interval $[x, y]$ in each case. This argument also holds if $a = -\infty$ or $b = \infty$, and also in the case of a closed or half-open interval. Similarly, if f is decreasing to the left of c and increasing to the right, then f is quasiconvex.

We could use this argument in Problem 2.5.2(d), for instance.

2.5.6 Since f is decreasing and g is increasing, it follows from Example 2.4.2 that both of these functions are quasiconcave as well as quasiconvex. Their sum, $f(x) + g(x) = x^3 - x$, is not quasiconcave,

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however. For instance, it is clear from Fig. A2.5.6 in the answer section of the book that the upper level set $P_0 = \{x : f(x) + g(x) \geq 0\} = [-1, 0] \cup [1, \infty)$, which is not a convex set of $\mathbb{R}$. Similarly, the lower level set $P^0 = (-\infty, -1] \cup [0, 1]$ is not convex, and so $f + g$ is not quasiconvex either.

2.5.7 (a) Let f be single-peaked with a peak at x^*. We want to prove that f is strictly quasiconcave. That is, we want to prove that if $x < y$ and z is a point strictly between x and y, then $f(z) > \min\{f(x), f(y)\}$. There are two cases to consider:

(A) Suppose $z \leq x^*$. Since $x < z \leq x^*$ and f is strictly increasing in the interval $[x, x^*]$, we have $f(z) > f(x)$.

(B) Suppose $x^* < z$. Then $x^* < z < y$, and since f is strictly decreasing in $[x^*, y]$, we get $f(z) > f(y)$.

In both cases we get $f(z) > \min\{f(x), f(y)\}$, and so f is strictly quasiconcave

(b) No. Even if f is concave, it may, for example, be linear in each of the intervals $(-\infty, x^*]$ and $[x^*, \infty)$, or in parts of one or both of these intervals, and in such cases f cannot be strictly concave.

2.5.11 With apologies to the reader, we would like to change the name of the function from f to h. So $h(\mathbf{x})$ is strictly quasiconcave and homogeneous of degree $q \in (0, 1)$, with $h(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in K and $h(\mathbf{0}) = 0$, and we want to prove that h is strictly concave in the convex cone K. Define a new function f by $f(\mathbf{x}) = h(\mathbf{x})^{1/q}$. Then $f(\mathbf{x})$ is also strictly quasiconcave (use the definition) and satisfies the conditions of Theorem 2.5.3. Let $\mathbf{x} \neq \mathbf{y}$ and let $\lambda \in (0, 1)$.

Assume first that $\mathbf{x}$ and $\mathbf{y}$ do not lie on the same ray from the origin in $\mathbb{R}^n$. Then both $f(\mathbf{x})$ and $f(\mathbf{y})$ are strictly positive and we can define α, β, μ, $\mathbf{x}'$, and $\mathbf{y}'$ as in the proof of Theorem 2.5.3. We get $\mathbf{x}' \neq \mathbf{y}'$ and $f(\mathbf{x}') = f(\mathbf{y}')$, and (see the original proof)

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= f(\mu\mathbf{x}' + (1-\mu)\mathbf{y}') > f(\mathbf{x}') = f(\mathbf{y}') = \mu f(\mathbf{x}') + (1-\mu)f(\mathbf{y}') \\ &= (\mu\beta/\alpha)f(\mathbf{x}) + (\beta - \beta\mu)f(\mathbf{y}) = \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \end{aligned}$$

with strict inequality because f is strictly quasiconcave. Since $0 < q < 1$, the qth power function $t \mapsto t^q$ is strictly increasing and strictly concave, and therefore

$$\begin{aligned} h(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= \big(f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})\big)^q > \big(\lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y})\big)^q \\ &> \lambda f(\mathbf{x})^q + (1-\lambda)f(\mathbf{y})^q = \lambda h(\mathbf{x}) + (1-\lambda)h(\mathbf{y}) \end{aligned}$$

It remains to show that $h(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) > \lambda h(\mathbf{x}) + (1-\lambda)h(\mathbf{y})$ in the case where $\mathbf{x}$ and $\mathbf{y}$ lie on the same ray from the origin. We can assume that $\mathbf{x} \neq 0$ and $\mathbf{y} = t\mathbf{x}$ for some nonnegative number $t \neq 1$. Since the qth power function is strictly concave, $(\lambda + (1-\lambda)t)^q > \lambda 1^q + (1-\lambda)t^q = \lambda + (1-\lambda)t^q$. It follows that

$$\begin{aligned} h(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= h\big((\lambda + (1-\lambda)t)\mathbf{x}\big) = \big(\lambda + (1-\lambda)t\big)^q h(\mathbf{x}) \\ &> \big(\lambda + (1-\lambda)t^q\big)h(\mathbf{x}) = \lambda h(\mathbf{x}) + (1-\lambda)h(t\mathbf{x}) = \lambda h(\mathbf{x}) + (1-\lambda)h(\mathbf{y}) \end{aligned}$$

2.6

2.6.1 (a) $f_1'(x, y) = ye^{xy}$, $f_2'(x, y) = xe^{xy}$, $f_{11}''(x, y) = y^2e^{xy}$. $f_{12}''(x, y) = e^{xy} + xye^{xy}$, $f_{22}''(x, y) = x^2e^{x,y}$. With the exception of f_{12}'', these derivatives all vanish at the origin, so we are left with the quadratic approximation $f(x, y) \approx f(0, 0) + f_{12}''(0, 0)xy = 1 + xy$.

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(b) $f_1'(x,y) = 2xe^{x^{-}y^2}$, $f_2'(x,y) = -2ye^{x^2-y^2}$, $f_{11}''(x,y) = (2+4x^2)e^{x^2-y^2}$, $f_{12}''(x,y) = -4xye^{x^2-y^2}$, $f_{22}''(x,y) = (-2+4y^2)e^{x^2-y^2}$. Hence, $f_1'(0,0) = f_2'(0,0) = 0$, $f_{11}''(0,0) = 2$, $f_{12}''(0,0) = 0$, $f_{22}''(0,0) = -2$.
Quadratic approximation: $f(x,y) \approx f(0,0) + \frac{1}{2}f_{11}''(0,0)x^2 + \frac{1}{2}f_{22}''(0,0)y^2 = 1 + x^2 - y^2$.

(c) The first- and second-order derivatives are: $f_1'(x,y) = \dfrac{1}{1+x+2y}$, $f_2'(x,y) = \dfrac{2}{1+x+2y}$, $f_{11}''(x,y) = -\dfrac{1}{(1+x+2y)^2}$, $f_{12}''(x,y) = -\dfrac{2}{(1+x+2y)^2}$, $f_{22}''(x,y) = -\dfrac{4}{(1+x+2y)^2}$.
At $(0,0)$ we get $f(0,0) = 1$, $f_1'(0,0) = 1$, $f_2'(0,0) = 2$, $f_{11}''(0,0) = -1$, $f_2''(0,0) = -2$, $f_{22}''(0,0) = -4$, and the quadratic approximation to $f(x,y)$ around $(0,0)$ is

$$\begin{aligned} f(x,y) &\approx f(0,0) + f_1'(0,0)x + f_2'(0,0)y + \tfrac{1}{2}f_{11}''(0,0)x^2 + f_{12}''(0,0)xy + \tfrac{1}{2}f_{22}''(0,0)y^2 \\ &= 1 + x + 2y - \tfrac{1}{2}x^2 - 2xy - 2y^2 \end{aligned}$$

(There is a misprint in the answer in the first printing of the book.)

2.6.4 z is defined implicitly as a function of x and y by the equation $F(x,y,z) = 0$, where $F(x,y,z) = \ln z - x^3y + xz - y$. With $x = y = 0$ we get $\ln z = 0$, so $z = 1$. Since $F_3'(x,y,z) = 1/z + x = 1 \neq 0$ at $(x,y,z) = (0,0,1)$, z is a C^1 function of x and y around this point. In order to find the Taylor polynomial we need the first- and second-order derivatives of z with respect to x and y. One possibility is to use the formula $z_x' = -F_1'(x,y,z)/F_3'(x,y,z)$ and the corresponding formula for z_y', and then take derivatives of these expressions to find the second-order derivatives. That procedure leads to some rather nasty fractions with a good chance of going wrong, so instead we shall differentiate the equation $\ln z = x^3y - xz + y$, *keeping in mind that z is a function of x and y*. We get

$$z_x'/z = 3x^2y - z - xz_x' \quad\Longrightarrow\quad (1+xz)z_x' = 3x^2yz - z^2 \tag{1}$$
$$z_y'/z = x^3 - xz_y' + 1 \quad\Longrightarrow\quad (1+xz)z_y' = x^3z + z \tag{2}$$

Taking derivatives with respect to x and y in (1), we get

$$(z + xz_x')z_x' + (1+xz)z_{xx}'' = 6xyz + 3x^2yz_x' - 2zz_x' \tag{3}$$
$$xz_y'z_x' + (1+xz)z_{xy}'' = 3x^2z + 3x^2yz_y' - 2zz_y' \tag{4}$$

and differentiating (2) with respect to y, we get

$$xz_y'z_y' + (1+xz)z_{yy}'' = x^3z_y' + z_y' \tag{5}$$

Now that we have finished taking derivatives, we can let $x = y = 0$ and $z = 1$ in the equations we have found. Equations (1) and (2) give $z_x' = -1$ and $z_y' = 1$ (at the particular point we are interested in), and then (3)–(5) give $z_{xx}'' = 3$, $z_{xy}'' = -2$, and $z_{yy}'' = 1$. The quadratic approximation to z around $(0,0)$ is therefore

$$z \approx 1 - x + y + \tfrac{3}{2}x^2 - 2xy + \tfrac{1}{2}y^2$$

(The first printing of the book has a misprint in the answer to this problem, too—the coefficient of x^2 is wrong.)

2.7

2.7.2 (a) $F(x,y,z) = x^3 + y^3 + z^3 - xyz - 1$ is obviously C^1 everywhere, and $F_3'(0,0,1) = 3 \neq 0$, so by the implicit function theorem the equation defines z as a C^1 function $g(x,y)$ in a neighbourhood

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of $(x_0, y_0, z_0) = (0, 0, 1)$. To find the partial derivatives g_1' and g_2' there is no need to use the matrix formulation in Theorem 2.7.2. After all, g is just a single real-valued function, and the partial derivative $g_1'(x, y)$ is just what we get if we treat y as a constant and take the derivative of g with respect to x. Thus, $g_1'(x, y) = -F_1'(x, y, z)/F_3'(x, y, z) = -(3x^2 - yz)/(3z^2 - xy)$ and, in particular, $g_1'(0, 0) = -F_1'(0, 0, 1)/F_3'(0, 0, 1) = 0$. Likewise, $g_2'(0, 0) = 0$.

(b) As in part (a), F_3' is C^1 everywhere and $F_3'(x, y, z) = e^z - 2z \neq 0$ for $z = 0$, so the equation $F = 0$ defines z as a C^1 function $g(x, y)$ around $(x_0, y_0, z_0) = (1, 0, 0)$. We get $g_1'(x, y) = -F_1'(x, y, z)/F_3'(x, y, z) = 2x/(e^z - 2z)$ and $g_2'(x, y, z) = -F_2'/F_3' = 2y/(e^z - 2z)$, so $g_1'(1, 0) = 2$ and $g_2'(1, 0) = 0$.

2.7.3 The given equation system can be written as $\mathbf{f}(x, y, z, u, v, w) = \mathbf{0}$, where $\mathbf{f} = (f_1, f_2, f_3)$ is the function $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_1(x, y, z, u, v, w) = y^2 - z + u - v - w^3 + 1$, $f_2(x, y, z, u, v, w) = -2x + y - z^2 + u + v^3 - w + 3$, and $f_3(x, y, z, u, v, w) = x^2 + z - u - v + w^3 - 3$. The Jacobian determinant of $\mathbf{f}$ with respect to u, v, w is

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & -1 & -3w^2 \\ 1 & 3v^2 & -1 \\ -1 & -1 & 3w^2 \end{vmatrix} = 6w^2 - 2$$

This determinant is different from 0 at P, so according to Theorem 2.7.2 the equation system does define u, v, and w as C^1 functions of x, y, and z. The easiest way to find the partial derivatives of these functions with respect to x is probably to take the derivatives with respect to x in each of the three given equations, remembering that u, v, w are functions of x, y, z. We get

$$\left.\begin{aligned} u_x' - v_x' - 3w^2 w_x' &= 0 \\ -2 + u_x' + 3v^2 v_x' - w_x' &= 0 \\ 2x - u_x' - v_x' + 3w^2 w_x' &= 0 \end{aligned}\right\}, \quad \text{so at } P \text{ we get} \quad \left\{\begin{aligned} u_x' - v_x' - 3w_x' &= 0 \\ u_x' \qquad - w_x' &= 2 \\ -u_x' - v_x' + 3w_x' &= -2 \end{aligned}\right.$$

The unique solution of this system is $u_x' = 5/2$, $v_x' = 1$, $w_x' = 1/2$. (In the first printing of the book w_x' was incorrectly given as $5/2$.)

2.7.4 The Jacobian determinant is $\begin{vmatrix} f_u' & f_v' \\ g_u' & g_v' \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}(\cos^2 v + \sin^2 v) = e^{2u}$, which is nonzero for all u (and v).

(a) $e^u \neq 0$, so the equations imply $\cos v = \sin v = 0$, but that is impossible because $\cos^2 v + \sin^2 v = 1$.

(b) We must have $\cos v = \sin v = e^{-u}$, and since $\cos^2 v + \sin^2 v = 1$ we get $2\cos^2 v = 1$ and therefore $\sin v = \cos v = \sqrt{1/2} = \frac{1}{2}\sqrt{2}$. (Remember that $\cos v = e^{-u}$ cannot be negative.) The only values of v that satisfy these equations are $v = (\frac{1}{4} + 2k)\pi$, where k runs through all integers. Further, $e^u = 1/\cos v = \sqrt{2}$ gives $u = \frac{1}{2}\sqrt{2}$.

2.7.6 The Jacobian is x_1. We find that $x_1 = y_1 + y_2$, $x_2 = y_2/(y_1 + y_2)$ (provided $y_1 + y_2 \neq 0$). The transformation maps the given rectangle onto the set S in the $y_1 y_2$-plane given by the inequalities

$$\text{(i)}\ \ 1 \leq y_1 + y_2 \leq 2, \qquad \text{(ii)}\ \ \frac{1}{2} \leq \frac{y_2}{y_1 + y_2} \leq \frac{2}{3}$$

The inequalities (i) show that $y_1 + y_2 > 0$, and if we multiply by $6(y_1 + y_2)$ in (ii) we get the equivalent inequalities

$$3(y_1 + y_2) \leq 6y_2 \leq 4(y_1 + y_2) \iff y_1 \leq y_2 \leq 2y_1 \tag{iii}$$

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It follows that S is a quadrilateral with corners at $(1/2, 1/2)$, $(1, 1)$, $(2/3, 4/3)$, and $(1/3, 2/3)$. See figure M2.7.6.

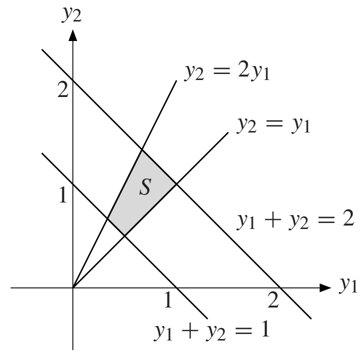

Figure M2.7.6

2.7.8 (a) $J = \begin{vmatrix} \dfrac{\partial}{\partial r}(r\cos\theta) & \dfrac{\partial}{\partial \theta}(r\cos\theta) \\ \dfrac{\partial}{\partial r}(r\sin\theta) & \dfrac{\partial}{\partial \theta}(r\sin\theta) \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

(b) $J \neq 0$ everywhere except at the origin in the $r\theta$-plane, so T is locally one-to-one in A. But it is not globally one-to-one in A, since we have, for example, $T(r, 0) = T(r, 2\pi)$.

2.7.10 (a) Taking differentials in the equation system

$$\begin{aligned} 1 + (x+y)u - (2+u)^{1+v} &= 0 \\ 2u - (1+xy)e^{u(x-1)} &= 0 \end{aligned} \tag{1}$$

we get

$$\begin{aligned} u(dx + dy) + (x+y)\,du - e^{(1+v)\ln(2+u)}\left[\ln(2+u)\,dv + \frac{1+v}{2+u}\,du\right] &= 0 \\ 2\,du - e^{u(x-1)}(y\,dx + x\,dy) - (1+xy)e^{u(x-1)}\big((x-1)\,du + u\,dx\big) &= 0 \end{aligned}$$

If we now let $x = y = u = 1$ and $v = 0$, we get

$$2\,du + dx + dy - 3\ln 3\,dv - du = 0$$

and

$$2\,du - dx - dy - 2\,dx = 0$$

Rearranging this system gives

$$\begin{aligned} du - 3\ln 3\,dv &= -dx - dy \\ 2\,du \qquad\qquad &= 3\,dx + dy \end{aligned}$$

with the solutions

$$du = \frac{3}{2}\,dx + \frac{1}{2}\,dy \qquad \text{and} \qquad dv = \frac{1}{3\ln 3}\left(\frac{5}{2}\,dx + \frac{3}{2}\,dy\right)$$

Hence,

$$u'_x(1,1) = \frac{\partial u}{\partial x}(1,1) = \frac{3}{2} \qquad \text{and} \qquad v'_x(1,1) = \frac{\partial v}{\partial x}(1,1) = \frac{5}{6\ln 3}$$

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(Alternatively we could have used the formula for the derivatives of an implicit function, but it is still a good idea to substitute the values of x, y, u, and v after finding the derivatives.)

(b) Define a function f by $f(u) = u - ae^{u(b-1)}$. Then $f(0) = -a$ and $f(1) = 1 - ae^{b-1}$. Since $b \leq 1$, we have $e^{b-1} \leq e^0 = 1$. It follows that $ae^{b-1} \leq a \leq 1$, so $f(1) \geq 0$. On the other hand, $f(0) \leq 0$, so the intermediate value theorem ensures that f has at least one zero in [0, 1]. Further, $f'(u) = 1 - a(b-1)e^{u(b-1)} \geq 1$, because $a(b-1) \leq 0$. Therefore f is strictly increasing and cannot have more than one zero, so the solution of $f(u) = 0$ is unique.

(c) For given values of x and y, let $a = (1+xy)/2$ and $b = x$. The equation in part (b) is then equivalent to the second equation in system (1). Thus we get a uniquely determined u, and this u belongs to [0, 1]. (Note that, when x and y lie in [0, 1], then the values of a and b that we have chosen also lie in [0, 1].) The first equation in (1) now determines v uniquely, as

$$v = -1 + \frac{\ln(1 + (x+y)u)}{\ln(2+u)}$$

2.8

2.8.1 (a) The system has 7 variables, Y, C, I, G, T, r, and M, and 4 equations, so the counting rule says that the system has $7 - 4 = 3$ degrees of freedom.

(b) We can write the system as

$$\begin{aligned} f_1(M,T,G,Y,C,I,r) &= Y - C - I - G = 0 \\ f_2(M,T,G,Y,C,I,r) &= C - f(Y-T) \ = 0 \\ f_3(M,T,G,Y,C,I,r) &= I - h(r) \qquad = 0 \\ f_4(M,T,G,Y,C,I,r) &= r - m(M) \qquad = 0 \end{aligned} \tag{$*$}$$

Suppose that f, h, and m are C^1 functions and that the system has an equilibrium point (i.e. a solution of the equations), $(M_0, T_0, G_0, Y_0, C_0, I_0, r_0)$. The Jacobian determinant of $\mathbf{f} = (f_1, f_2, f_3, f_4)$ with respect to (Y, C, I, r) is

$$\frac{\partial(f_1, f_2, f_3, f_4)}{\partial(Y, C, I, r)} = \begin{vmatrix} 1 & -1 & -1 & 0 \\ -f'(Y-T) & 1 & 0 & 0 \\ 0 & 0 & 1 & -h'(r) \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 - f'(Y-T)$$

so by the implicit function theorem the system $(*)$ defines Y, C, I, and r as C^1 functions of M, T, and G around the equilibrium point if $f'(Y-T) \neq 1$.

Note that the functions h and m have no influence on the Jacobian determinant. The reason is that once M, T, and G are given, the last equation in $(*)$ immediately determines r, and the next to last equation then determines I. The problem therefore reduces to the question whether the first two equations can determine Y and T when the values of the other variables are given. The implicit function theorem tells us that the answer will certainly be yes if the Jacobian determinant

$$\left|\frac{\partial(f_1, f_2)}{\partial(Y, C)}\right| = \begin{vmatrix} 1 & -1 \\ -f'(Y-T) & 1 \end{vmatrix} = 1 - f'(Y-T)$$

is nonzero, i.e. if $f'(Y-T) \neq 1$.

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2.8.2 (a) The Jacobian determinants are

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} e^{x+y} & e^{x+y} \\ 4x+4y-1 & 4x+4y-1 \end{vmatrix} = 0 \tag{i}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} \dfrac{1}{y} & \dfrac{-x}{y^2} \\ \dfrac{-2y}{(y+x)^2} & \dfrac{2x}{(x+y)^2} \end{vmatrix} = \frac{2x}{y(y+x)^2} - \frac{2x}{y(y+x)^2} = 0 \tag{ii}$$

(b) (i) It is not hard to see that $v = 2(x+y)^2 - (x+y)$ and $x + y = \ln u$, so $v = 2(\ln u)^2 - \ln u$.

(ii) We have $x = uy$, so

$$v = \frac{y - uy}{y + uy} = \frac{1-u}{1+u}$$

2.8.3 We need to assume that the functions f and g are C^1 in an open ball around (x_0, y_0). For all (x, y) in A the Jacobian determinant is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} f'_1(x,y) & f'_2(x,y) \\ g'_1(x,y) & g'_2(x,y) \end{vmatrix} = 0$$

so $f'_2(x,y)g'_1(x,y) = f'_1(x,y)g'_2(x,y)$. Since $f'_1(x_0,y_0) \neq 0$, the equation $G(x,y,u) = u - f(x,y) = 0$ defines x as a function $x = \varphi(y,u)$ in an open ball around (y_0, u_0), where $u_0 = f(x_0, y_0)$. Within this open ball we have $\varphi'_1(y,u) = \partial x/\partial y = -(\partial G/\partial y)/(\partial G/\partial x) = -f'_2(x,y)/f'_1(x,y)$. Moreover, $v = g(x,y) = g(\varphi(y,u),y)$. If we let $\psi(y,u) = g(\varphi(y,u),y)$, then $\psi'_1(y,u) = g'_1\varphi'_1 + g'_2 = -f'_2 g'_1/f'_1 + g'_2 = -f'_1 g'_2/f'_1 + g'_2 = 0$. Thus, $\psi(y,u)$ only depends on u, and $v = \psi(y,u)$ is a function of u alone. Hence, v is functionally dependent on u.

(If we let $F(u,v) = v - \psi(y_0,u)$, then F satisfies the requirements of definition (2.8.5).)

Chapter 3 Static Optimization

3.1

3.1.3 (a) The first-order conditions for maximum are

$$\frac{1}{3}pv_1^{-2/3}v_2^{1/2} - q_1 = 0, \qquad \frac{1}{2}pv_1^{1/3}v_2^{-1/2} - q_2 = 0$$

with the unique solution

$$v_1^* = \frac{1}{216}p^6 q_1^{-3} q_2^{-3}, \qquad v_2^* = \frac{1}{144}p^6 q_1^{-2} q_2^{-4}$$

The objective function $pv_1^{1/3}v_2^{1/2} - q_1v_1 - q_2v_2$ is concave, cf. the display (2.5.6) on Cobb–Douglas functions, and therefore (v_1^*, v_2^*) is a maximum point.

(b) The value function is

$$\pi^*(p, q_1, q_2) = p(v_1^*)^{1/3}(v_2^*)^{1/2} - q_1v_1^* - q_2v_2^* = \frac{1}{432}p^6 q_1^{-2} q_2^{-3}$$

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and it follows that

$$\frac{\partial \pi^*(p, q_1, q_2)}{\partial p} = \frac{1}{72} p^5 q_1^{-2} q_2^{-3} = (v_1^*)^{1/3} (v_2^*)^{1/2}$$

$$\frac{\partial \pi^*(p, q_1, q_2)}{\partial q_1} = -\frac{1}{216} p^6 q_1^{-3} q_2^{-3} = -v_1^*, \qquad \frac{\partial \pi^*(p, q_1, q_2)}{\partial q_2} = -\frac{1}{144} p^6 q_1^{-2} q_2^{-4} = -v_2^*$$

3.1.5 The first-order conditions for maximum are

$$f_1'(x, y, r, s) = r^2 - 2x = 0, \qquad f_2'(x, y, r, s) = 3s^2 - 16y = 0$$

with the solutions $x^*(r, s) = \frac{1}{2}r^2$ and $y^*(r, s) = \frac{3}{16}s^2$. Since $f(x, y, r, s)$ is concave with respect to (x, y), the point (x^*, y^*) is a maximum point. Moreover,

$$f^*(r, s) = f(x^*, y^*, r, s) = \frac{1}{4}r^4 + \frac{9}{32}s^4$$

so

$$\frac{\partial f^*(r, s)}{\partial r} = r^3, \qquad \frac{\partial f^*(r, s)}{\partial s} = \frac{9}{8}s^3$$

On the other hand,

$$\frac{\partial f(x, y, r, s)}{\partial r} = 2rx, \qquad \frac{\partial f(x, y, r, s)}{\partial s} = 6sy$$

so

$$\left[\frac{\partial f(x, y, r, s)}{\partial r}\right]_{(x,y)=(x^*,y^*)} = 2rx^* = r^3, \qquad \left[\frac{\partial f(x, y, r, s)}{\partial s}\right]_{(x,y)=(x^*,y^*)} = 6sy^* = \frac{9}{8}s^3$$

in accordance with the envelope result (3.1.3).

3.1.6 We want to maximize

$$\pi(\mathbf{v}, p, \mathbf{q}, \mathbf{a}) = \pi(v_1, \dots, v_n; p, q_1, \dots, q_n, a_1, \dots, a_n) = \sum_{i=1}^{n} (pa_i \ln(v_i + 1) - q_i v_i)$$

with respect to $v_1, \dots, v_n$ for given values of p, $\mathbf{q} = (q_1, \dots, q_n)$, and $\mathbf{a} = (a_1, \dots, a_n)$. Since $\partial\pi/\partial v_i = pa_i/(v_i + 1) - q_i$, the only stationary point is $\mathbf{v}^* = \mathbf{v}^*(p, \mathbf{q}, \mathbf{a}) = (v_1^*, \dots, v_n^*)$, where $v_i^* = pa_i/q_i - 1$. Since π is concave with respect to $v_1, \dots, v_n$, this is a maximum point. The corresponding maximum value is

$$\begin{aligned}\pi^*(p, q_1, \dots, q_n, a_1, \dots, a_n) &= \sum_{i=1}^{n} (pa_i \ln(v_i^* + 1) - q_i v_i^*) = \sum_{i=1}^{n} \Big(pa_i \ln\big(\frac{pa_i}{q_i}\big) - q_i\big(\frac{pa_i}{q_i} - 1\big)\Big) \\ &= \sum_{i=1}^{n} (pa_i \ln p + pa_i \ln a_i - pa_i \ln q_i - pa_i + q_i)\end{aligned}$$

Easy calculations now yield

$$\frac{\partial \pi^*(p, \mathbf{q}, \mathbf{a})}{\partial p} = \sum_{i=1}^{n} a_i \ln\big(\frac{pa_i}{q_i}\big) = \sum_{i=1}^{n} a_i \ln(v_i^* + 1) = \left[\frac{\partial \pi(\mathbf{v}, p, \mathbf{q}, \mathbf{a})}{\partial p}\right]_{\mathbf{v}=\mathbf{v}^*(p,\mathbf{q},\mathbf{a})}$$

$$\frac{\partial \pi^*(p, \mathbf{q}, \mathbf{a})}{\partial q_i} = -\frac{pa_i}{q_i} + 1 = -v_i^* = \left[\frac{\partial \pi(\mathbf{v}, p, \mathbf{q}, \mathbf{a})}{\partial q_i}\right]_{\mathbf{v}=\mathbf{v}^*(p,\mathbf{q},\mathbf{a})}$$

$$\frac{\partial \pi^*(p, \mathbf{q}, \mathbf{a})}{\partial a_i} = p \ln\big(\frac{pa_i}{q_i}\big) = p \ln(v_i^* + 1) = \left[\frac{\partial \pi(\mathbf{v}, p, \mathbf{q}, \mathbf{a})}{\partial a_i}\right]_{\mathbf{v}=\mathbf{v}^*(p,\mathbf{q},\mathbf{a})}$$

in accordance with formula (3.1.3).

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3.2

3.2.1 The first-order conditions are

$$
\begin{aligned}
f_1'(x_1, x_2, x_3) &= 2x_1 - x_2 + 2x_3 = 0 \\
f_2'(x_1, x_2, x_3) &= -x_1 + 2x_2 + x_3 = 0 \\
f_3'(x_1, x_2, x_3) &= 2x_1 + x_2 + 6x_3 = 0
\end{aligned}
$$

The determinant of this linear equation system is 4, so by Cramer's rule it has a unique solution. This solution is of course $(x_1, x_2, x_3) = (0, 0, 0)$. The Hessian matrix is (at every point)

$$
\mathbf{H} = \mathbf{f}''(x_1, x_2, x_3) = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{pmatrix}
$$

with leading principal minors $D_1 = 2$, $D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$, and $D_3 = |\mathbf{H}| = 4$, so $(0, 0, 0)$ is a local minimum point by Theorem 3.2.1(a).

3.2.3 (a) The first-order conditions are

$$
\begin{aligned}
f_1'(x, y, z) &= 2x + 2xy = 0 \\
f_2'(x, y, z) &= x^2 + 2yz = 0 \\
f_3'(x, y, z) &= y^2 + 2z - 4 = 0
\end{aligned}
$$

This system gives five stationary points: $(0, 0, 2)$, $(0, \pm 2, 0)$, $(\pm\sqrt{3}, -1, 3/2)$. The Hessian matrix is

$$
\mathbf{f}''(x, y, z) = \begin{pmatrix} 2 + 2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 2 \end{pmatrix}
$$

with leading principal minors $D_1 = 2 + 2y$, $D_2 = 4(1 + y)z - 4x^2$, and $D_3 = 8(1 + y)(z - y^2) - 8x^2$. The values of the leading principal minors at the stationary points are given in the following table:

	D_1	D_2	D_3
$(0, 0, 2)$	2	8	16
$(0, 2, 0)$	6	0	-96
$(0, -2, 0)$	-2	0	32
$(\sqrt{3}, -1, 3/2)$	0	-12	-24
$(-\sqrt{3}, -1, 3/2)$	0	-12	-24

It follows from Theorem 3.2.1 that $(0, 0, 2)$ is a local minimum point, and all the other stationary points are saddle points.

(b) The stationary points are the solutions of the equation system

$$
\begin{aligned}
f_1'(x_1, x_2, x_3, x_4) &= 8x_2 - 8x_1 = 0 \\
f_2'(x_1, x_2, x_3, x_4) &= 20 + 8x_1 - 12x_2^2 = 0 \\
f_3'(x_1, x_2, x_3, x_4) &= 48 - 24x_3 = 0 \\
f_4'(x_1, x_2, x_3, x_4) &= 6 - 2x_4
\end{aligned}
$$

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The first equation gives $x_2 = x_1$, and then the second equation gives $12x_1^2 - 8x_1 - 20 = 0$ with the two solutions $x_1 = 5/3$ and $x_1 = -1$. The last two equations determine x_3 and x_4. There are two stationary points, $(5/3, 5/3, 2, 3)$ and $(-1, -1, 2, 3)$. The Hessian matrix is

$$\mathbf{f}''(x_1, x_2, x_3, x_4) = \begin{pmatrix} -8 & 8 & 0 & 0 \\ 8 & -24x_2 & 0 & 0 \\ 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

and the leading principal minors of the Hessian are

$$D_1 = -8, \quad D_2 = 192x_2 - 64 = 64(3x_2 - 1), \quad D_3 = -24D_2, \quad D_4 = 48D_2$$

At $(-1, -1, 2, 3)$ we get $D_2 < 0$, so this point is a saddle point. The other stationary point, $(5/3, 5/3, 2, 3)$, we get $D_1 < 0$, $D_2 > 0$, $D_3 < 0$, and $D_4 > 0$, so this point is a local maximum point.

3.3

3.3.2 (a) The admissible set is the intersection of the sphere $x^2+y^2+z^2 = 216$ and the plane $x+2y+3z = 0$, which passes through the center of the sphere. This set (a circle) is closed and bounded (and nonempty!), and by the extreme value theorem the objective function does attain a maximum over the admissible set. The Lagrangian is $\mathcal{L}(x, y, z) = x + 4y + z - \lambda_1(x^2 + y^2 + z^2 - 216) - \lambda_2(x + 2y + 3z)$, and the first-order conditions are:

$$\text{(i)}\ \ 1 - 2\lambda_1 x - \lambda_2 = 0, \quad \text{(ii)}\ \ 4 - 2\lambda_1 y - 2\lambda_2 = 0, \quad \text{(iii)}\ \ 1 - 2\lambda_1 z - 3\lambda_2 = 0$$

From (i) and (ii) we get $\lambda_2 = 1 - 2\lambda_1 x = 2 - \lambda_1 y$, which implies $\lambda_1(y - 2x) = 1$. Conditions (i) and (iii) yield $1 - 2\lambda_1 x = \frac{1}{3} - \frac{2}{3}\lambda_1 z = 0$, which implies $\lambda_1(\frac{2}{3}z - 2x) = -\frac{2}{3}$. Multiply by $-\frac{3}{2}$ to get $\lambda_1(3x - z) = 1$.

It follows that $y - 2x = 3x - z$, so $z = 5x - y$. Inserting this expression for z in the constraint $x + 2y + 3z = 0$ yields $16x - y = 0$, so $\underline{y = 16x}$ and $\underline{z = -11x}$. The constraint $x^2 + y^2 + z^2 = 216$ then yields $(1 + 256 + 121)x^2 = 216$, so $x^2 = 216/378 = 4/7$, and $x = \pm\frac{2}{7}\sqrt{7}$. Hence there are two points that satisfy the first-order conditions:

$$\mathbf{x}^1 = \left(\tfrac{2}{7}\sqrt{7}, \tfrac{32}{7}\sqrt{7}, -\tfrac{22}{7}\sqrt{7}\right), \qquad \mathbf{x}^2 = \left(-\tfrac{2}{7}\sqrt{7}, -\tfrac{32}{7}\sqrt{7}, \tfrac{22}{7}\sqrt{7}\right)$$

The multipliers are then $\lambda_1 = 1/(y - 2x) = 1/(14x) = \pm\frac{1}{28}\sqrt{7}$ and $\lambda_2 = 1 - 2\lambda_1 x = \frac{6}{7}$.

The objective function, $f(x, y, z) = x + 4y + z$, attains it maximum value $f_{\max} = \frac{108}{7}\sqrt{7}$ at $\mathbf{x}^1$, with $\lambda_1 = \frac{1}{28}\sqrt{7}$, $\lambda_2 = \frac{6}{7}$. (The point $\mathbf{x}^2$ is the minimum point and $f_{\min} = -\frac{108}{7}\sqrt{7}$.)

Comment: It is clear that the Lagrangian is concave if $\lambda_1 > 0$ and convex if $\lambda_1 < 0$. Therefore part (b) of Theorem 3.3.1 is sufficient to show that $\mathbf{x}^1$ is a maximum point and $\mathbf{x}^2$ is a minimum point in this problem, so we did not need to use the extreme value theorem.

(b) Equation (3.3.10) shows that $\Delta f^* \approx \lambda_1 \cdot (-1) + \lambda_2 \cdot 0.1 = -\frac{1}{28}\sqrt{7} + \frac{0.6}{7} \approx -0.009$.

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3.3.4 (a) With the Lagrangian $\mathcal{L}(x, y) = \frac{1}{2}\ln(1 + x_1) + \frac{1}{4}\ln(1 + x_2) - \lambda(2x_1 + 3x_2 - m)$, the first-order conditions are

$$\text{(i)}\ \frac{1}{2(1+x_1)} - 2\lambda = 0, \qquad \text{(ii)}\ \frac{1}{4(1+x_2)} - 3\lambda = 0$$

Equations (i) and (ii) yield

$$\lambda = \frac{1}{4(1+x_1)} = \frac{1}{12(1+x_2)} \implies 1 + x_1 = 3(1 + x_2) \iff x_1 - 3x_2 = 2$$

Together with the constraint $2x_1 + 3x_2 = m$ this gives $x_1 = x_1^*(m) = \frac{1}{3}(m+2)$, $x_2 = x_2^*(m) = \frac{1}{9}(m-4)$, and then $\lambda = \frac{3}{4}(m+5)^{-1}$.

(b) $U^*(m) = \frac{1}{2}\ln(1+x_1^*(m)) + \frac{1}{4}\ln(1+x_2^*(m)) = \frac{1}{2}\ln(\frac{1}{3}(m+5)) + \frac{1}{4}\ln(\frac{1}{9}(m+5)) = \frac{3}{4}\ln(m+5) - \ln 3$, so $dU^*/dm = \frac{3}{4}(m+5)^{-1} = \lambda$.

3.3.6 (a) The two constraints determine an ellipsoid centred at the origin and a plane through the origin, respectively. The admissible set is the curve of intersection of these two surfaces, namely an ellipse. This curve is closed and bounded (and nonempty), so the extreme value theorem guarantees the existence of both maximum and minimum points. It is not very hard to show that the matrix $\mathbf{g}'(\mathbf{x})$ in Theorem 3.3.1 has rank 2 at all admissible points, so the usual first-order conditions are necessary.

Lagrangian: $\mathcal{L}(x, y, z) = x^2 + y^2 + z^2 - \lambda_1(x^2 + y^2 + 4z^2 - 1) - \lambda_2(x + 3y + 2z)$. First-order conditions:

$$\text{(i)}\ 2x - 2\lambda_1 x - \lambda_2 = 0, \qquad \text{(ii)}\ 2y - 2\lambda_1 y - 3\lambda_2 = 0, \qquad \text{(iii)}\ 2z - 8\lambda_1 z - 2\lambda_2 = 0$$

From (i) and (ii) we get (iv) $\lambda_2 = 2(1 - \lambda_1)x = \frac{2}{3}(1 - \lambda_1)y$.

(A) If $\lambda_1 = 1$, then $\lambda_2 = 0$ and (iii) implies $z = 0$. The constraints reduce to $x^2 + y^2 = 1$ and $x + 3y = 0$, and we get the two candidates

$$(x, y, z) = \left(\pm\tfrac{3}{10}\sqrt{10}\,, \mp\tfrac{1}{10}\sqrt{10}\,, 0\right) \qquad (*)$$

(B) If $\lambda_1 \neq 1$, then (iv) implies $y = 3x$, and the second constraint gives $z = -5x$. The first constraint then yields $x^2 + 9x^2 + 100x^2 = 1$ which leads to the two candidates

$$(x, y, z) = \left(\pm\tfrac{1}{110}\sqrt{110}, \pm\tfrac{3}{110}\sqrt{110}\,, \mp\tfrac{5}{110}\sqrt{110}\right) \qquad (**)$$

In this case the multipliers λ_1 and λ_2 can be determined from equations (i) and (iii), and we get $\lambda_1 = \frac{7}{22}$ and $\lambda_2 = \frac{15}{11}x$.

The objective function, $x^2 + y^2 + z^2$, attains its maximum value 1 at the points $(*)$, while the points $(**)$ give the minimum value 7/22. It is worth noting that with $\lambda_1 = 1$ the Lagrangian is concave (linear, in fact), so Theorem 3.3.1(b) shows that the points in $(*)$ are maximum points, even if we do not check the rank condition in Theorem 3.3.1(a) (but without that rank condition we cannot be sure that the first-order conditions are necessary, so there might be other maximum points beside the two that we have found).

(There is a much simpler way to find the maximum points in this problem: Because of the first constraint, the objective function $x^2 + y^2 + z^2$ equals $1 - 3z^2$, which obviously has a maximum for $z = 0$. We then just have to solve the equations $x^2 + y^2 = 1$ and $x + 3y = 0$ for x and y.)

(b) $\Delta f^* \approx 1 \cdot 0.05 + 0 \cdot 0.05 = 0.05$.

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3.3.10 There was a misprint in this problem in the first printing of the book: The constraints should all be equality constraints, so please correct $g_j(\mathbf{x}, \mathbf{r}) \leq 0$ to $g_j(\mathbf{x}, \mathbf{r}) = 0$ for $j = 1, \ldots, m$.

Now consider two problems, namely the problem in (3.3.13) and the problem

$$\max f(\mathbf{x}, \mathbf{r}) \text{ subject to } \begin{cases} g_j(\mathbf{x}, \mathbf{r}) = 0, & j = 1, \ldots, m \\ r_i = b_{m+i}, & i = 1, \ldots, k \end{cases} \tag{$*$}$$

The Lagrangian for problem (3.3.13) is $\mathcal{L}(\mathbf{x}, \mathbf{r}) = f(\mathbf{x}, \mathbf{r}) - \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}, \mathbf{r})$ and the Lagrangian for problem $(*)$ is $\widetilde{\mathcal{L}}(\mathbf{x}, \mathbf{r}) = f(\mathbf{x}, \mathbf{r}) - \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}, \mathbf{r}) - \sum_{i=1}^{k} \lambda_{m+i}(r_i - b_{m+i})$. The first-order conditions for maximum in problem $(*)$ imply that

$$\frac{\partial \widetilde{\mathcal{L}}(\mathbf{x}, \mathbf{r})}{\partial r_i} = \frac{\partial f(\mathbf{x}, \mathbf{r})}{\partial r_i} - \sum_{j=1}^{m} \frac{\partial g_j(\mathbf{x}, \mathbf{r})}{\partial r_i} - \lambda_{m+i} = 0, \qquad i = 1, \ldots, k$$

Equation (3.3.9) implies that $\partial f^*(\mathbf{r})/\partial r_i = \lambda_{m+i}$, and so

$$\frac{\partial f^*(\mathbf{r})}{\partial r_i} = \lambda_{m+i} = \frac{\partial f(\mathbf{x}, \mathbf{r})}{\partial r_i} - \sum_{j=1}^{m} \frac{\partial g_j(\mathbf{x}, \mathbf{r})}{\partial r_i} = \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{r})}{\partial r_i}$$

3.4

3.4.3 Lagrangian: $\mathcal{L}(x, y, z) = x + y + z - \lambda_1(x^2 + y^2 + z^2 - 1) - \lambda_2(x - y - z - 1)$. First-order conditions:

$$\text{(i)}\ 1 - 2\lambda_1 x - \lambda_2 = 0, \quad \text{(ii)}\ 1 - 2\lambda_1 y + \lambda_2 = 0, \quad \text{(iii)}\ 1 - 2\lambda_1 z + \lambda_2 = 0$$

Equations (ii) and (iii) give $2\lambda_1 y = 2\lambda_1 z$. If $\lambda_1 = 0$, then (i) and (ii) yield $1 - \lambda_2 = 0$ and $1 + \lambda_2 = 0$. This is clearly impossible, so we must have $\lambda_1 \neq 0$, and therefore $y = z$. We solve this equation together with the two constraints, and get the two solutions

$$\begin{aligned} (x_1, y_1, z_1) &= (1, 0, 0), & \lambda_1 &= 1, & \lambda_2 &= -1 \\ (x_2, y_2, z_2) &= (-\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}), & \lambda_1 &= -1, & \lambda_2 &= \tfrac{1}{3} \end{aligned}$$

In the second-derivative test (Theorem 3.4.1) we now have $n = 3$ and $m = 2$, so all we need check is

$$B_3(x, y, z) = \begin{vmatrix} 0 & 0 & 2x & 2y & 2z \\ 0 & 0 & 1 & -1 & -1 \\ 2x & 1 & -2\lambda_1 & 0 & 0 \\ 2y & -1 & 0 & -2\lambda_1 & 0 \\ 2z & -1 & 0 & 0 & -2\lambda_1 \end{vmatrix}$$

A little tedious computation yields $B_3(x_1, y_1, z_1) = -16$ and $B_3(x_2, y_2, z_2) = 16$. Hence (x_1, y_1, z_1) is a local maximum point and (x_2, y_2, z_2) is a local minimum point.

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3.5

3.5.2 (a) The Lagrangian is $\mathcal{L}(x,y) = \ln(x+1) + \ln(y+1) - \lambda_1(x+2y-c) - \lambda_2(x+y-2)$, and the necessary Kuhn–Tucker conditions for (x,y) to solve the problem are:

$$\mathcal{L}_1'(x,y) = \frac{1}{x+1} - \lambda_1 - \lambda_2 = 0 \tag{1}$$

$$\mathcal{L}_2'(x,y) = \frac{1}{y+1} - 2\lambda_1 - \lambda_2 = 0 \tag{2}$$

$$\lambda_1 \geq 0, \quad \text{and } \lambda_1 = 0 \text{ if } x + 2y < c \tag{3}$$

$$\lambda_2 \geq 0, \quad \text{and } \lambda_2 = 0 \text{ if } x + y < 2 \tag{4}$$

Of course, (x,y) must also satisfy the constraints

$$x + 2y \leq c \tag{5}$$

$$x + y \leq 2 \tag{6}$$

(b) Let $c = 5/2$. We consider the four possible combinations of $\lambda_1 = 0$, $\lambda_1 > 0$, $\lambda_2 = 0$, and $\lambda_2 > 0$.

(I) $\underline{\lambda_1 = \lambda_2 = 0}$. This contradicts (1), so no candidates.

(II) $\underline{\lambda_1 > 0,\ \lambda_2 = 0}$. From (3) and (5), $x+2y = 5/2$. Moreover, by eliminating λ_1 from (1) and (2) we get $x+1 = 2y+2$. The last two equations have the solution $x = 7/4$, $y = 3/8$. But then $x+y = 17/8 > 2$, contradicting (6). No candidates.

(III) $\underline{\lambda_1 = 0,\ \lambda_2 > 0}$. From (4) and (6), $x+y = 2$. Moreover, eliminating λ_2 from (1) and (2) we get $x = y$, and so $x = y = 1$. But then $x + 2y = 3 > 5/2$, contradicting (5). No candidates.

(IV) $\underline{\lambda_1 > 0,\ \lambda_2 > 0}$. Then $x+y = 2$ and $x+2y = 5/2$, so $x = 3/2$, $y = 1/2$. We find that $\lambda_1 = 4/15$ and $\lambda_2 = 2/15$, so this is a candidate, and the only one.

The Lagrangian is obviously concave in x and y, so $(x,y) = (3/2, 1/2)$ solves the problem.

(c) If we assume that $V(c)$ is a differentiable function of c, then formula (3.5.6) yields $V'(5/2) = \lambda_1 = 4/15$.

A direct argument can run as follows: For all values of c such that λ_1 and λ_2 are positive, x and y must satisfy the constraints with equality, and so $x = 4 - c$ and $y = c - 2$. Then equations (1) and (2) yield

$$\lambda_1 = \frac{1}{y+1} - \frac{1}{x+1} = \frac{1}{c-1} - \frac{1}{5-c} \qquad \text{and} \qquad \lambda_2 = \frac{2}{x+1} - \frac{1}{y+1} = \frac{2}{5-c} - \frac{1}{c-1}$$

It is clear that these expressions remain positive for c in an open interval around $c = 5/2$. (More precisely, they are both positive if and only if $7/3 < c < 3$.) For such c, the derivative of the value function is

$$V'(c) = \frac{\partial V}{\partial x}\frac{dx}{dc} + \frac{\partial V}{\partial y}\frac{dy}{dc} = -\frac{1}{1+x} + \frac{1}{1+y} = \lambda_1$$

3.5.3 We reformulate the problem as a standard maximization problem:

$$\text{maximize } -4\ln(x^2+2) - y^2 \quad \text{subject to} \quad \begin{cases} -x^2 - y \leq -2 \\ -x \leq -1 \end{cases}$$

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The Lagrangian is $\mathcal{L}(x, y) = -4\ln(x^2+2) - y^2 - \lambda_1(-x^2 - y + 2) - \lambda_2(-x+1)$, so the necessary Kuhn–Tucker conditions together with the constraints are:

$$\mathcal{L}_1' = -\frac{8x}{x^2+2} + 2\lambda_1 x + \lambda_2 = 0 \tag{i}$$

$$\mathcal{L}_2' = -2y + \lambda_1 = 0 \tag{ii}$$

$$\lambda_1 \geq 0, \quad \text{and } \lambda_1 = 0 \text{ if } x^2 + y > 2 \tag{iii}$$

$$\lambda_2 \geq 0, \quad \text{and } \lambda_2 = 0 \text{ if } x > 1 \tag{iv}$$

$$x^2 + y \geq 2 \tag{v}$$

$$x \geq 1 \tag{vi}$$

We try the four possible combinations of zero or positive multipliers:

(A) $\underline{\lambda_1 = 0,\ \lambda_2 = 0}$. From (i) we see that $x = 0$, which contradicts $x \geq 1$.

(B) $\underline{\lambda_1 = 0,\ \lambda_2 > 0}$. From (iv) and (vi), $x = 1$, and (ii) gives $y = 0$. This contradicts (v).

(C) $\underline{\lambda_1 > 0,\ \lambda_2 = 0}$. From (iii) and (v) we get $x^2 + y = 2$. Equation (i) gives $\lambda_1 = 4/(x^2+2)$, and then (ii) gives $y = \lambda_1/2 = 2/(x^2+2)$. Inserted into $x^2+y = 2$, this gives $x^4 = 2$, or $x = \sqrt[4]{2}$. It follows that $y = 2/(\sqrt{2}+2) = 2 - \sqrt{2}$, and $\lambda_1 = 4 - 2\sqrt{2}$. So $(x, y, \lambda_1, \lambda_2) = (\sqrt[4]{2}, 2-\sqrt{2}, 4-2\sqrt{2}, 0)$ is a candidate.

(D) $\underline{\lambda_1 > 0,\ \lambda_2 > 0}$. Then (iii)–(vi) imply $x^2 + y = 2$ and $x = 1$. So $x = y = 1$. Then from (i), $\lambda_2 + 2\lambda_1 = 8/3$ and (ii) gives $\lambda_1 = 2$. But then $\lambda_2 = 8/3 - 4 < 0$. Contradiction.

Thus the only possible solution is the one given in (C), and the minimum value of $f(x, y) = 4\ln(x^2+2) + y^2$ under the given constraints is $f(\sqrt[4]{2}, 2-\sqrt{2}) = 4\ln(\sqrt{2}+2) + (2-\sqrt{2})^2 = 4\ln(\sqrt{2}+2) + 6 - 4\sqrt{2}$.

3.5.4 With the Lagrangian $\mathcal{L}(x, y) = -(x-a)^2 - (y-b)^2 - \lambda_1(x-1) - \lambda_2(y-2)$ we get the Kuhn–Tucker necessary conditions for (x, y) to be a maximum point:

$$\mathcal{L}_1'(x, y) = -2(x-a) - \lambda_1 = 0 \tag{i}$$

$$\mathcal{L}_2'(x, y) = -2(y-b) - \lambda_2 = 0 \tag{ii}$$

$$\lambda_1 \geq 0, \quad \text{and } \lambda_1 = 0 \text{ if } x < 1 \tag{iii}$$

$$\lambda_2 \geq 0, \quad \text{and } \lambda_2 = 0 \text{ if } y < 2 \tag{iv}$$

Of course, a maximum point (x, y) must also satisfy the constraints (v) $x \leq 1$ and (vi) $y \leq 2$.

We try the four possible combinations of $\lambda_1 = 0$, $\lambda_1 > 0$, $\lambda_2 = 0$, and $\lambda_2 > 0$.

(A) $\underline{\lambda_1 = 0,\ \lambda_2 = 0}$. Equations (i) and (ii) give $x = a$ and $y = b$. Because of the constraints this is possible only if $a \leq 1$ and $b \leq 2$.

(B) $\underline{\lambda_1 = 0,\ \lambda_2 > 0}$. We get $x = a$ and $y = 2$. Constraint (v) implies $a \leq 1$, and equation (ii) yields $b = y + \frac{1}{2}\lambda_2 > y = 2$.

(C) $\underline{\lambda_1 > 0,\ \lambda_2 = 0}$. This gives $x = 1$, $y = b$, $a = 1 + \frac{1}{2}\lambda_1 > 1$, and $b \leq 2$.

(D) $\underline{\lambda_1 > 0,\ \lambda_2 > 0}$. With both multipliers positive, both constraints must be satisfied with equality: $x = 1$, $y = 2$. We also get $a = 1 + \frac{1}{2}\lambda_1 > 1$ and $b = 2 + \frac{1}{2}\lambda_2 > 2$.

We observe that in each of the four cases (A)–(D) there is exactly one point that satisfies the Kuhn–Tucker conditions, and since the objective function is concave, these points are maximum points. Which case applies depends on the values of a and b. The solution can be summarized as $x^* = \min\{a, 1\}$, $y^* = \min\{b, 2\}$.

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The admissible set in this problem is the same as in Example 3.5.1, and the maximization problem is equivalent to finding an admissible point as close to (a, b) as possible. It is readily seen that the point (x^*, y^*) given above is the optimal point. In particular, in case (A) the point (a, b) itself is admissible and is also the optimal point.

3.5.6 (a) With the Lagrangian $\mathcal{L}(x, y) = x^5 - y^3 - \lambda_1(x - 1) - \lambda_2(x - y)$ the Kuhn–Tucker conditions and the constraints are

$$
\begin{aligned}
\mathcal{L}_1'(x, y) &= 5x^4 - \lambda_1 - \lambda_2 = 0 && \text{(i)}\\
\mathcal{L}_2'(x, y) &= \quad -3y^2 + \lambda_2 = 0 && \text{(ii)}\\
\lambda_1 &\ge 0, \quad \text{and } \lambda_1 = 0 \text{ if } x < 1 && \text{(iii)}\\
\lambda_2 &\ge 0, \quad \text{and } \lambda_2 = 0 \text{ if } x < y && \text{(iv)}\\
x &\le 1 && \text{(v)}\\
x &\le y && \text{(vi)}
\end{aligned}
$$

Consider the four possible combinations of zero or positive multipliers:

(A) $\lambda_1 = 0,\ \lambda_2 = 0$. Equations (i) and (ii) give $x = 0$ and $y = 0$. Thus $(x_1, y_1) = (0, 0)$ is a candidate for optimum.

(B) $\lambda_1 = 0,\ \lambda_2 > 0$. Since $\lambda_2 > 0$, the complementary slackness condition (iv) tells us that we cannot have $x < y$, while constraint (vi) says $x \le y$. Therefore $x = y$. Equations (i) and (ii) then give $5x^4 = 0 + \lambda_2 = 3y^2$, and thus $5y^4 = 3y^2$. Since $\lambda_2 \ne 0$ we have $y \ne 0$, and therefore $5y^2 = 3$, so $x = y = \pm\sqrt{3/5} = \pm\sqrt{15/25} = \pm\frac{1}{5}\sqrt{15}$. We get two new candidates, $(x_2, y_2) = (\frac{1}{5}\sqrt{15}, \frac{1}{5}\sqrt{15})$ and $(x_3, y_3) = (-\frac{1}{5}\sqrt{15}, -\frac{1}{5}\sqrt{15})$.

(C) $\lambda_1 > 0,\ \lambda_2 = 0$. Now (ii) gives $y = 0$, while (iii) and (v) give $x = 1$. But this violates constraint (vi), so we get no new candidates for optimum here.

(D) $\lambda_1 > 0,\ \lambda_2 > 0$. The complementary slackness conditions show that in this case both constraints must be satisfied with equality, so we get one new candidate point, $(x_4, y_4) = (1, 1)$

Evaluating the objective function $h(x, y) = x^5 - y^3$ at each of the four maximum candidates we have found shows that

$$
\begin{aligned}
h(x_1, y_1) &= h(x_4, y_4) = 0\\
h(x_2, y_2) &= x_2^5 - y_2^3 = x_2^5 - x_2^3 = (x_2^2 - 1)x_2^3 = -\tfrac{2}{5}x_2^3 = -\tfrac{6}{125}\sqrt{15}\\
h(x_3, y_3) &= h(-x_2, -y_2) = -h(x_2, y_2) = \tfrac{6}{125}\sqrt{15}
\end{aligned}
$$

(For the last evaluation we used the fact that h is an odd function, i.e. $h(-x, -y) = -h(x, y)$.) Hence, *if* there is a maximum point for h in the feasible set, then (x_3, y_3) is that maximum point. Note that although the feasible set is closed, it is not bounded, so it is not obvious that there is a maximum. But part (b) of this problem will show that a maximum point does exist.

(b) Let $h(x, y) = x^5 - y^3$ and define $f(x) = \max_{y \ge x} h(x, y)$. Since $h(x, y)$ is strictly decreasing with respect to y, it is clear that we get the maximum when $y = x$, so $f(x) = h(x, x) = x^5 - x^3$. To find the maximum of $f(x)$ for $x \le 1$ we take a look at

$$f'(x) = 5x^2 - 3x^2 = 5x^2(x^2 - \tfrac{3}{5}) = 5x^2(x + x_2)(x - x_2)$$

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where $x_2 = \sqrt{3/5} = \frac{1}{5}\sqrt{15}$ (as in part (a)). It is clear that

$$f'(x) > 0 \text{ if } x \in (-\infty, -x_2) \text{ or } x \in (x_2, \infty)$$
$$f'(x) < 0 \text{ if } x \in (-x_2, 0) \text{ or } x \in (0, x_2)$$

Therefore f is strictly increasing in the interval $(-\infty, -x_2]$, strictly decreasing in $[-x_2, x_2]$, and strictly increasing again in $[x_2, \infty)$. It follows that $f(x)$ will reach its highest value when $x = -x_2$ or when $x = 1$. (Draw a graph!) Since $f(-x_2) = \frac{6}{125}$ and $f(1) = 0$, the maximum point is $-x_2$ ($= x_3$).

So why does this show that the point (x_3, y_3) that we found in part (a) really is a maximum point in that problem? The reason is that for every point (x, y) with $x \leq 1$ and $x \leq y$ we have

$$h(x, y) \leq f(x) \leq f(x_3) = h(x_3, x_3) = h(x_3, y_3)$$

3.6

3.6.2 Lagrangian: $\mathcal{L}(x, y) = xy + x + y - \lambda_1(x^2 + y^2 - 2) - \lambda_2(x + y - 1)$. First-order conditions:

$$\mathcal{L}_1'(x, y) = y + 1 - 2\lambda_1 x - \lambda_2 = 0 \tag{i}$$
$$\mathcal{L}_2'(x, y) = x + 1 - 2\lambda_1 y - \lambda_2 = 0 \tag{ii}$$
$$\lambda_1 \geq 0, \quad \text{and } \lambda_1 = 0 \text{ if } x^2 + y^2 < 2 \tag{iii}$$
$$\lambda_2 \geq 0, \quad \text{and } \lambda_2 = 0 \text{ if } x + y < 1 \tag{iv}$$

It is usually a good idea to exploit similarities and symmetries in the first-order conditions. In the present case, we can eliminate λ_2 from (i) and (ii) to get

$$y - 2\lambda_1 x = x - 2\lambda_1 y \iff (1 + 2\lambda_1)(y - x) = 0$$

Since $1 + 2\lambda_1 \geq 1$, this implies $y = x$ for any point that satisfies the first-order conditions. Now consider the various combinations of zero or positive values of λ_1 and λ_2:

(A) $\underline{\lambda_1 = 0,\ \lambda_2 = 0}$. Equations (i) and (ii) give $(x, y) = (-1, -1)$.

(B) $\underline{\lambda_1 = 0,\ \lambda_2 > 0}$. Since $\lambda_2 > 0$, we must have $x + y = 1$, and since $x = y$, we get $(x, y) = (1/2, 1/2)$.

(C) $\underline{\lambda_1 > 0,\ \lambda_2 = 0}$. Now $x^2 + y^2 = 2$ because $\lambda_1 > 0$, and since $x = y$ we get $x = y = \pm 1$. The point $(1, 1)$ violates the constraint $x + y \leq 1$. If $x = y = -1$, then equation (i) yields $2\lambda_1 = \lambda_2 = 0$, contradicting the assumption $\lambda_1 > 0$. Thus there are no candidate points in this case. (We did get the point $(-1, -1)$ in case (A) above.)

(D) $\underline{\lambda_1 > 0,\ \lambda_2 > 0}$. In this case both constraints must be active, i.e. $x^2 + y^2 = 1$ and $x + y = 1$. Since $x = y$, the first constraint yields $x = y = 1/2$, but then $x^2 + y^2 \neq 2$. So no candidates in this case either.

Comparing the values of the objective function $xy + x + y$ at the two points $(-1, -1)$ and $(1/2, 1/2)$ shows that $(1/2, 1/2)$ must be the maximum point. (The extreme value theorem guarantees that there really is a maximum point.)

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3.7

3.7.1 (a) The Lagrangian is $\mathcal{L}(x, y) = 100 - e^{-x} - e^{-y} - e^{-z} - \lambda_1(x + y + z - a) - \lambda_2(x - b)$ and the Kuhn–Tucker conditions are

$$
\begin{aligned}
e^{-x} - \lambda_1 - \lambda_2 &= 0 && \text{(i)}\\
e^{-y} - \lambda_1 \qquad &= 0 && \text{(ii)}\\
e^{-z} - \lambda_1 \qquad &= 0 && \text{(iii)}\\
\lambda_1 \geq 0, \quad \text{and } \lambda_1 &= 0 \text{ if } x + y + z < a && \text{(iv)}\\
\lambda_2 \geq 0, \quad \text{and } \lambda_2 &= 0 \text{ if } x < b && \text{(v)}
\end{aligned}
$$

Equations (ii) and (iii) imply that $\lambda_1 > 0$ and $y = z$. From (iv) we get $x + y + z = a$, so $x + 2y = a$.

(A) Suppose $\underline{\lambda_2 = 0}$. Then (i) and (ii) imply $\lambda_1 = e^{-x}$ and $x = y$. Hence $x + y + z = 3x = a$, so $x = a/3$, and therefore $a/3 \leq b$, i.e. $a \leq 3b$.

(B) Suppose $\underline{\lambda_2 > 0}$. Condition (v) now implies $x = b$, and so $y = (a - x)/2 = (a - b)/2$. Then $\lambda_1 = e^{-y} = e^{-(a-b)/2}$, and (i) yields $\lambda_2 = e^{-b} - e^{-(a-b)/2}$. Since $\lambda_2 > 0$, we must have $-b > -(a-b)/2$, i.e. $a > 3b$.

Thus, the Kuhn–Tucker conditions have a unique solution in each of the two cases $a \leq 3b$ and $a > 3b$. The Lagrangian is concave, so we know that the points we have found really are optimal.

(b) See the answer in the book for the evaluation of $\partial f^*(a, b)/\partial a$ and $\partial f^*(a, b)/\partial b$. Strictly speaking, if $a = 3b$, you must consider the one-sided derivatives and show that the left and right derivatives are the same, but things work out all right in that case too.

3.7.3 (a) Consider the maximization problem and write it in standard form as

$$
\text{maximize} \quad f(x, y) = x^2 + y^2 \quad \text{subject to} \quad \begin{cases} 2x^2 + 4y^2 \leq s^2 & (*)\\ -2x^2 - 4y^2 \leq -r^2 & (**) \end{cases}
$$

We use the Lagrangian $\mathcal{L}(x, y, r, s) = x^2 + y^2 - \lambda_1(2x^2 + 4y^2 - s^2) + \lambda_2(2x^2 + 4y^2 - r^2)$, and get the Kuhn–Tucker conditions

$$
\begin{aligned}
\mathcal{L}'_1 &= 2x - 4\lambda_1 x + 4\lambda_2 x = 0 && \text{(i)}\\
\mathcal{L}'_2 &= 2y - 8\lambda_1 y + 8\lambda_2 y = 0 && \text{(ii)}\\
\lambda_1 \geq 0, &\quad \text{and} \quad \lambda_1 = 0 \text{ if } 2x^2 + 4y^2 < s^2 && \text{(iii)}\\
\lambda_2 \geq 0, &\quad \text{and} \quad \lambda_2 = 0 \text{ if } 2x^2 + 4y^2 > r^2 && \text{(iv)}
\end{aligned}
$$

If $\lambda_1 = 0$, then (i) and (ii) would yield

$$
(2 + 4\lambda_2)x = 0 \implies x = 0, \qquad (2 + 8\lambda_2)y = 0 \implies y = 0
$$

which contradicts the constraint $2x^2 + 4y^2 \geq r^2$. Therefore we must have $\lambda_1 > 0$ and $2x^2 + 4y^2 = s^2$. Moreover, we cannot have $\lambda_2 > 0$, for that would imply $2x^2 + 4y^2 = r^2 < s^2$, which contradicts $2x^2 + 4y^2 = s^2$. Hence, $\underline{\lambda_1 > 0,\ \lambda_2 = 0}$. Equations (i) and (ii) reduce to

$$
\text{(i}'\text{)} \quad (2 - 4\lambda_1)x = 0 \qquad \text{and} \qquad \text{(ii}'\text{)} \quad (2 - 8\lambda_1)y = 0
$$

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If $x = 0$, then $y \neq 0$ (because $2x^2 + 4y^2 = s^2 > 0$), and (ii′) implies $\lambda_1 = 1/4$. If $x \neq 0$, then (i′) implies $\lambda_1 = 1/2$. We are left with the two possibilities

$$\text{(A)}\ \ \lambda_1 = 1/2,\ y = 0,\ x = \pm\tfrac{1}{2}\sqrt{2}\,s \qquad \text{(B)}\ \ \lambda_1 = 1/4,\ x = 0,\ y = \pm\tfrac{1}{2}s$$

Case (A) gives the optimum points, $(x^*, y^*) = (\pm\frac{1}{2}\sqrt{2}\,s, 0)$, and $f^*(r, s) = f(x^*, y^*) = \frac{1}{2}s^2$. To verify the envelope result (3.7.5), note that

$$\partial\mathcal{L}(x, y, r, s)/\partial r = -2\lambda_2 r, \qquad \partial\mathcal{L}(x, y, r, s)/\partial s = 2\lambda_1 s$$

If we insert the optimal values of x^* and y^* and the corresponding values $\lambda_1 = 1/2$ and $\lambda_2 = 0$ of the multipliers, we get

$$\partial\mathcal{L}(x^*, y^*, r, s)/\partial r = 0 = \partial f^*(r, s)/\partial r, \qquad \partial\mathcal{L}(x^*, y^*, r, s)/\partial s = s = \partial f^*(r, s)/\partial s$$

in accordance with (3.7.5).

(b) The minimization problem is equivalent to maximizing $g(x, y) = -f(x, y)$ subject to the constraints (∗) and (∗∗) in part (a). We get a new Lagrangian $\overline{\mathcal{L}}(x, y, r, s) = -x^2 - y^2 - \lambda_1(2x^2 + 4y^2 - s^2) + \lambda_2(2x^2 + 4y^2 - r^2)$, and the first-order conditions are as in (a), except that (i) and (ii) are replaced by

$$\overline{\mathcal{L}}_1' = -2x - 4\lambda_1 x + 4\lambda_2 x = 0 \qquad \text{(i}'\text{)}$$
$$\overline{\mathcal{L}}_2' = -2y - 8\lambda_1 y + 8\lambda_2 y = 0 \qquad \text{(ii}'\text{)}$$

The solution proceeds along the same lines as in (a), but this time $\lambda_2 = 0$ is impossible, so we get $\lambda_2 > 0$ and $\lambda_1 = 0$. The optimum points are $(x^*, y^*) = (0, \pm\frac{1}{2}r)$, with $\lambda_2 = 1/4$, and $g^*(r, s) = g(x^*, y^*) = -(x^*)^2 - (y^*)^2 = -\frac{1}{4}r^2$, and the minimum value of f is $f^*(r, s) = -g^*(r, s) = \frac{1}{4}r^2$. The equations in (3.7.5) now become

$$\overline{\mathcal{L}}_r'(x^*, y^*, r, s) = -2\lambda_2 r = -\tfrac{1}{2}r = \frac{\partial g^*}{\partial r} = -\frac{\partial f^*}{\partial r}, \quad \overline{\mathcal{L}}_s'(x^*, y^*, r, s) = 2\lambda_1 s = 0 = \frac{\partial g^*}{\partial s} = -\frac{\partial f^*}{\partial s}$$

(c) The admissible set is the area between two ellipses, and the problems in (a) and (b) are equivalent to finding the largest and the smallest distance from the origin to a point in this admissible set.

3.7.4 Let $\mathbf{r}$ and $\mathbf{s}$ be points in the domain of f^*, let $\lambda \in [0, 1]$, and put $\mathbf{t} = \lambda\mathbf{r} + (1-\lambda)\mathbf{s}$. We want to prove that $f^*(\mathbf{t}) \geq \lambda f^*(\mathbf{r}) + (1-\lambda)f^*(\mathbf{s})$. There are points $\mathbf{x}$ and $\mathbf{y}$ such that $f^*(\mathbf{r}) = f(\mathbf{x}, \mathbf{r})$ and $f^*(\mathbf{s}) = f(\mathbf{y}, \mathbf{s})$. Let $\mathbf{w} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$. Since $\mathbf{g}$ is convex, we have $\mathbf{g}(\mathbf{w}, \mathbf{t}) \leqq \lambda\mathbf{g}(\mathbf{x}, \mathbf{r}) + (1-\lambda)\mathbf{g}(\mathbf{y}, \mathbf{s}) \leqq \mathbf{0}$, so $(\mathbf{w}, \mathbf{t})$ is admissible. And since f is concave, we have

$$\begin{aligned} f^*(\mathbf{t}) \geq f(\mathbf{w}, \mathbf{t}) &= f(\lambda(\mathbf{x}, \mathbf{r}) + (1-\lambda)(\mathbf{y}, \mathbf{s})) \\ &\geq \lambda f(\mathbf{x}, \mathbf{r}) + (1-\lambda)f(\mathbf{y}, \mathbf{s}) = \lambda f^*(\mathbf{r}) + (1-\lambda)f^*(\mathbf{s}) \end{aligned}$$

3.8

3.8.2 (a) Lagrangian: $\mathcal{L}(x, y) = xy - \lambda(x + 2y - 2)$. Kuhn–Tucker conditions:

$$\partial\mathcal{L}/\partial x = y - \lambda \leq 0 \quad (= 0 \text{ if } x > 0) \qquad \text{(i)}$$
$$\partial\mathcal{L}/\partial y = x - 2\lambda \leq 0 \quad (= 0 \text{ if } y > 0) \qquad \text{(ii)}$$
$$\lambda \geq 0 \quad (= 0 \text{ if } x + 2y < 2) \qquad \text{(iii)}$$

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There are admissible points where $xy > 0$, so we cannot have $x = 0$ or $y = 0$ at the optimum point or points. It follows that (i) and (ii) must be satisfied with equality, and λ must be positive. Hence, $x = 2\lambda = 2y$ and $x + 2y = 2$, so $x = 1$, $y = 1/2$, $\lambda = 1/2$. (The extreme value theorem guarantees that there is a maximum, since the admissible set is closed and bounded. It is the closed line segment between (2, 0) and (0, 1).)

(b) As in part (a), there are admissible points where $x^\alpha y^\beta > 0$, so we may just as well accept (2, 0) and (0, 1) as admissible points and replace the constraints $x > 0$ and $y > 0$ by $x \geq 0$ and $y \geq 0$. Then the extreme value theorem guarantees that there is a maximum point in this case too, and it is clear that both x and y must be positive at the optimum. With the Lagrangian $\mathcal{L}(x, y) = x^\alpha y^\beta - \lambda(x + 2y - 2)$ we get the Kuhn–Tucker conditions

$$\partial\mathcal{L}/\partial x = \alpha x^{\alpha-1} y^\beta - \lambda \leq 0 \quad (= 0 \text{ if } x > 0) \tag{i}$$
$$\partial\mathcal{L}/\partial y = \beta x^\alpha y^{\beta-1} - 2\lambda \leq 0 \quad (= 0 \text{ if } y > 0) \tag{ii}$$
$$\lambda \geq 0 \quad (= 0 \text{ if } x + 2y < 2) \tag{iii}$$

It is clear that (i), (ii), and (iii) must all be satisfied with equality, and that $\lambda > 0$. From (i) and (ii) we get $\beta x^\alpha y^{\beta-1} = 2\lambda = 2\alpha x^{\alpha-1} y^\beta$, so $\beta x = 2\alpha y$. This equation, combined with (iii) yields the solution: $x = 2\alpha/(\alpha + \beta)$, $y = \beta/(\alpha + \beta)$.

(If we had not extended the admissible set to include the end points, then we could not have used the extreme value theorem to guarantee a maximum, but with the conditions on α and β the Lagrangian is concave, so we could still be certain that the point we have found is a maximum point. But the argument above, with a closed and bounded admissible set, works for all positive values of α and β, even if $\mathcal{L}$ is not concave.)

3.8.3 With the Lagrangian $\mathcal{L}(x, y, c) = cx + y - \lambda(x^2 + 3y^2 - 2)$ we get the Kuhn–Tucker conditions:

$$\mathcal{L}'_1 = c - 2\lambda x \leq 0 \quad (= 0 \text{ if } x > 0) \tag{i}$$
$$\mathcal{L}'_2 = 1 - 6\lambda y \leq 0 \quad (= 0 \text{ if } y > 0) \tag{ii}$$
$$\lambda \geq 0 \quad (= 0 \text{ if } x^2 + 3y^2 < 2) \tag{iii}$$

If $\lambda = 0$, then (ii) implies $1 \leq 0$, but that is impossible. Hence, $\lambda > 0$ and $x^2 + 3y^2 = 2$. Further, (ii) implies $6\lambda y \geq 1$, so $y > 0$. Therefore (ii) is an equality and $y = 1/6\lambda$.

(A) If $x = 0$, then (i) implies $c \leq 2\lambda x = 0$. Further, $3y^2 = 2$, so $y = \sqrt{2/3} = \sqrt{6}/3$, and $\lambda = 1/6y = \sqrt{6}/12$.

(B) If $x > 0$, then (i) is satisfied with equality and $c = 2\lambda x > 0$, and $x = c/2\lambda$. The equation $x^2 + 3y^2 = 2$ then leads to $\lambda = \sqrt{6(3c^2 + 1)}/12$, $x = 6c/\sqrt{6(3c^2 + 1)}$, and $y = 2/\sqrt{6(3c^2 + 1)}$.

Since the admissible set is closed and bounded and the objective function $f(x, y) = cx + y$ is continuous, the extreme value theorem guarantees that there is a maximum point for every value of c. The cases (A) and (B) studied above show that the Kuhn–Tucker conditions have exactly one solution in each case, so the solutions we found above are the optimal ones.

If $c \leq 0$, then we are in case (A) and $f^*(c) = cx^* + y^* = \sqrt{6}/3$.

If $c > 0$, then we are in case (B) and $f^*(c) = cx^* + y^* = \sqrt{6(3c^2 + 1)}/3$.

The value function $f^*(c)$ is obviously continuous for $c \neq 0$, and because $\lim_{c\to 0^+} f^*(c) = \sqrt{6}/3 = f^*(0) = \lim_{c\to 0^-} f^*(c)$, it is continuous at $c = 0$ too. The value function is differentiable at all $c \neq 0$,

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and it is not hard to show that both one-sided derivatives of $f^*(c)$ at $c = 0$ are 0, so f^* is differentiable there too.

For $c \le 0$ we get $(f^*)'(c) = 0 = x^*$, and a little calculation shows that $(f^*)'(c) = x^*$ for all $c > 0$ as well. Thus $(f^*)'(c) = \mathcal{L}_3'(x^*, y^*, c)$ for all c in accordance with equation (3.7.5).

3.8.5 See Fig. A3.8.5 in the answer section of the book. Since the Lagrangian

$$\mathcal{L}(x, y) = x + y - \tfrac{1}{2}(x+y)^2 - \tfrac{1}{4}x - \tfrac{1}{3}y - \lambda_1(x-5) - \lambda_2(y-3) - \lambda_3(-x+2y-2)$$

is concave, a point that satisfies Kuhn–Tucker conditions must be a maximum point. The objective function has no stationary points, so any maximum points must lie on the boundary of S. The Kuhn–Tucker conditions are:

$$\begin{aligned}
\mathcal{L}_1'(x, y) = 1 - (x+y) - \tfrac{1}{4} - \lambda_1 + \lambda_3 &\le 0 \quad (= 0 \text{ if } x > 0) &\text{(i)}\\
\mathcal{L}_2'(x, y) = 1 - (x+y) - \tfrac{1}{3} - \lambda_2 - 2\lambda_3 &\le 0 \quad (= 0 \text{ if } y > 0) &\text{(ii)}\\
\lambda_1 &\ge 0 \quad (= 0 \text{ if } x < 5) &\text{(iii)}\\
\lambda_2 &\ge 0 \quad (= 0 \text{ if } y < 3) &\text{(iv)}\\
\lambda_3 &\ge 0 \quad (= 0 \text{ if } -x + 2y < 2) &\text{(v)}
\end{aligned}$$

The solution is $x = 3/4$, $y = 0$, with $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Once we have found or been told about this point it is easy to check that it satisfies (i)–(v), but otherwise it can be a very tedious job to go through all possible combinations of zero or positive multipliers as well as $x = 0$ or $x > 0$ and $y = 0$ or $y > 0$. In this problem there are 32 different combinations to check if we do not see any shortcuts. In fact it would probably be more efficient to check each of the five straight line segments that form the boundary of S. But it would hardly be practical to do that in a problem with more than two variables, because it quickly becomes difficult to visualize the geometry of the admissible set.

3.8.6 (a) The last inequality in $(*)$ gives

$$\sum_{j=1}^{m} \lambda_j^*(g_j(\mathbf{x}^*) - b_j) \ge \sum_{j=1}^{m} \lambda_j(g_j(\mathbf{x}^*) - b_j) \quad \text{for all} \quad \boldsymbol{\lambda} \geqq \mathbf{0} \tag{$**$}$$

If $g_k(\mathbf{x}^*) > b_k$ for some k, then $\sum_{j=1}^m \lambda_j(g_j(\mathbf{x}^*) - b_j)$ can be made arbitrary large by choosing λ_k large and $\lambda_j = 0$ for all $j \ne k$. Hence, $g_j(\mathbf{x}^*) \le b_j$, $j = 1, \dots, m$. By choosing all λ_j equal to 0 in $(**)$, we get $\sum_{j=1}^m \lambda_j^*(g_j(\mathbf{x}^*) - b_j) \ge 0$. Now, $\lambda_j^* \ge 0$ and $g_j(\mathbf{x}^*) \le b_j$ for every j, so each $\lambda_j^*(g_j(\mathbf{x}^*) - b_j)$ must be zero, and then $\sum_{j=1}^m \lambda_j^*(g_j(\mathbf{x}^*) - b_j) = 0$. Finally, whenever $\mathbf{x}$ is admissible, the inequality $\widehat{\mathcal{L}}(\mathbf{x}, \boldsymbol{\lambda}^*) \le \widehat{\mathcal{L}}(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ implies that $f(\mathbf{x}) - f(\mathbf{x}^*) \le \sum_{j=1}^m \lambda_j^*[g_j(\mathbf{x}) - g_j(\mathbf{x}^*)] = \sum_{j=1}^m \lambda_j^*[g_j(\mathbf{x}) - b_j] \le 0$. Therefore $f(\mathbf{x}) \le f(\mathbf{x}^*)$, so $\mathbf{x}^*$ solves problem (1).

(b) Proof of the second inequality in $(*)$: Under the given assumptions, $\widehat{\mathcal{L}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) - \widehat{\mathcal{L}}(\mathbf{x}^*, \boldsymbol{\lambda}) = \sum_{j=1}^m \lambda_j[g_j(\mathbf{x}^*) - b_j] - \sum_{j=1}^m \lambda_j^*[g_j(\mathbf{x}^*) - b_j] = \sum_{j=1}^m \lambda_j[g_j(\mathbf{x}^*) - b_j] \le 0$ when $\boldsymbol{\lambda} \ge 0$. Since the first inequality in $(*)$ is assumed, we have shown that $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is a saddle point for $\widehat{\mathcal{L}}$.

3.9

3.9.1 (A) implies that $\pi(\mathbf{x}^*) \ge \pi(\hat{\mathbf{x}})$, i.e. $f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j g_j(\mathbf{x}^*) \ge f(\hat{\mathbf{x}}) - \sum_{j=1}^m \lambda_j g_j(\hat{\mathbf{x}})$. But, because $\hat{\mathbf{x}}$ also solves (3.9.1), $f(\hat{\mathbf{x}}) = f(\mathbf{x}^*)$ and then $\sum_{j=1}^m \lambda_j g_j(\hat{\mathbf{x}}) \ge \sum_{j=1}^m \lambda_j g_j(\mathbf{x}^*)$. Thus, because $\lambda_j \ge 0$ and

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$g_j(\hat{\mathbf{x}}) \le b_j$, $j = 1, \dots, m$, and also because of (3.9.5), we have

$$\sum_{j=1}^{m} \lambda_j b_j \ge \sum_{j=1}^{m} \lambda_j g_j(\hat{\mathbf{x}}) \ge \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}^*) = \sum_{j=1}^{m} \lambda_j b_j \tag{$*$}$$

Here the two middle terms, being squeezed between two equal numbers, must themselves be equal. Therefore $f(\hat{\mathbf{x}}) - \sum_{j=1}^{m} \lambda_j g_j(\hat{\mathbf{x}}) = f(\mathbf{x}^*) - \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}^*) \ge f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x})$ for all $\mathbf{x} \geqq \mathbf{0}$, proving (A). Also, if $g_k(\hat{\mathbf{x}}) < b_k$ and $\lambda_k > 0$ for any k, then $\sum_{j=1}^{m} \lambda_j (g_j(\hat{\mathbf{x}}) - b_j) < 0$, which contradicts $(*)$. Thus $\hat{\mathbf{x}}$ satisfies (A)–(C).

3.10

3.10.1 (a) Since the admissible set is closed and bounded, there is at least one maximum point in this problem. We use Theorem 3.8.3 (necessary first-order conditions for problems with mixed constraints). With $\mathcal{L}(x, y, z, a) = x^2 + y^2 + z^2 - \lambda(2x^2 + y^2 + z^2 - a^2) - \mu(x + y + z)$, the necessary conditions are:

$$\partial\mathcal{L}/\partial x = 2x - 4\lambda x - \mu = 0 \iff 2(1 - 2\lambda)x = \mu \tag{i}$$
$$\partial\mathcal{L}/\partial y = 2y - 2\lambda y - \mu = 0 \iff 2(1 - \lambda)y = \mu \tag{ii}$$
$$\partial\mathcal{L}/\partial z = 2z - 2\lambda z - \mu = 0 \iff 2(1 - \lambda)z = \mu \tag{iii}$$
$$\lambda \ge 0, \quad \text{and } \lambda = 0 \text{ if } 2x^2 + y^2 + z^2 < a^2 \tag{iv}$$

There is no sign restriction on μ, since the corresponding constraint is an equality, not an inequality.

If $\lambda = 0$, we get $x = y = z = \mu/2$, which implies $3x = 0$, so $x = y = z = \mu = 0$. But the point $(x, y, z) = (0, 0, 0)$ is obviously not a maximum point but a global minimum point in this problem.

If $\lambda > 0$, then $2x^2 + y^2 + z^2 = a^2$ because of complementary slackness. There are two possibilities: $\lambda = 1$ and $\lambda \ne 1$.

(A) If $\lambda \ne 1$ then (ii) and (iii) imply $y = z$, so the constraints yield $x + 2y = 0$ and $2x^2 + 2y^2 = a^2$, with the solutions $(x, y, z) = (\pm\frac{1}{5}\sqrt{10}\,a, \mp\frac{1}{10}\sqrt{10}\,a, \mp\frac{1}{10}\sqrt{10}\,a)$. Since $x = -2y$, equations (i) and (ii) yield $-4(1 - 2\lambda)y = \mu = 2(1 - \lambda)y$, so $\lambda = 3/5$.

(B) If $\lambda = 1$, then $\mu = 0$ and $x = 0$, so $y + z = 0$ and $y^2 + z^2 = a^2$, with $y = -z = \sqrt{a^2/2}$. This gives the two points $(x, y, z) = (0, \pm\frac{1}{2}\sqrt{2}\,a, \mp\frac{1}{2}\sqrt{2}\,a)$.

If we evaluate the objective function $x^2 + y^2 + z^2$ at each of the points we have found, we find that the two points $(x^*, y^*, z^*) = (0, \pm\frac{1}{2}\sqrt{2}\,a, \mp\frac{1}{2}\sqrt{2}\,a)$, with $\lambda = 1$, $\mu = 0$ both solve the problem. We have found several other points that satisfy the necessary conditions for a maximum but are not maximum points. Such is often the case when the Lagrangian is not concave.

(b) $f^*(a) = a^2$, so $df^*(a)/da = 2a$, and $\partial\mathcal{L}(x^*, y^*, z^*, a)/\partial a = 2\lambda a = 2a$.

3.11

3.11.1 Assume first that f is a function of just one variable, and let x_0 be any point in $A \subseteq \mathbb{R}$. There are three cases to consider:

(I) If $f(x_0) > 0$, then $f(x) > 0$ for all x in an open interval U around x_0 (because f is continuous). Then, for all x in U we have $f^+(x)^2 = f(x)^2$, and so $(d/dx)(f^+(x)^2) = 2f(x)f'(x) = 2f^+(x)f'(x)$.

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(II) If $f(x_0) < 0$, then there is an open interval V around x_0 such that for all x in V we have $f(x) < 0$ and $f^+(x) = 0$, and then $(d/dx)(f^+(x)^2) = 0 = 2f^+(x)f'(x)$.

(III) (The most interesting case.) If $f(x_0) = 0$, let $K = |f'(x_0)|$. There is then an open interval W around x_0 such that for all $x \neq x_0$ in W,

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < 1, \quad \text{which implies} \quad \left|\frac{f(x) - f(x_0)}{x - x_0}\right| < |f'(x_0)| + 1 = K + 1$$

and therefore

$$|f(x)| = |f(x) - f(x_0)| < (K+1)|x - x_0|$$

Then for all $x \neq x_0$ in W, we get

$$\left|\frac{f^+(x)^2 - f^+(x_0)^2}{x - x_0}\right| = \left|\frac{f^+(x)^2}{x - x_0}\right| \leq \left|\frac{f(x)^2}{x - x_0}\right| < (K+1)^2|x - x_0| \to 0 \quad \text{as} \quad x \to x_0$$

and so $((f^+)^2)'(x_0) = 0 = 2f^+(x_0)f'(x_0)$.

Thus, in all three cases we have shown that $(d/dx)(f^+(x)^2) = 2f^+(x)f'(x)$. This result immediately carries over to the partial derivatives of $f^+(\mathbf{x})^2$ if f is a function $\mathbb{R}^n \to \mathbb{R}$:

$$\frac{\partial}{\partial x_i}(f^+(\mathbf{x}))^2 = 2f^+(\mathbf{x})\frac{\partial}{\partial x_i}f(\mathbf{x})$$

and the gradient of $f^+(\mathbf{x})^2$ is

$$\begin{aligned}\nabla(f^+(\mathbf{x})^2) &= \Big(\frac{\partial}{\partial x_1}(f^+(\mathbf{x}))^2, \dots, \frac{\partial}{\partial x_n}(f^+(\mathbf{x}))^2\Big) \\ &= 2f^+(\mathbf{x})\Big(\frac{\partial}{\partial x_1}f(\mathbf{x}), \dots, \frac{\partial}{\partial x_n}f(\mathbf{x})\Big) = 2f^+(\mathbf{x})\nabla f(\mathbf{x})\end{aligned}$$

Note that f need not really be C^1. All that is needed is that all the first-order partial derivatives of f exist.

Chapter 4 Topics in Integration

4.1

4.1.5 (a) Expand the integrand. We get

$$\int_4^9 \frac{(\sqrt{x} - 1)^2}{x}\,dx = \int_4^9 \frac{x - 2\sqrt{x} + 1}{x}\,dx = \int_4^9 \Big(1 - \frac{2}{\sqrt{x}} + \frac{1}{x}\Big)\,dx = \Big|_4^9 (x - 4\sqrt{x} + \ln x) = 1 + \ln\frac{9}{4}$$

(b) With $u = 1 + \sqrt{x}$ we get $x = (u-1)^2$, $dx = 2(u-1)\,du$, and

$$\begin{aligned}\int_0^1 \ln(1 + \sqrt{x})\,dx &= \int_1^2 2(u-1)\ln u\,du = \Big|_1^2 (u-1)^2 \ln u - \int_1^2 \frac{(u-1)^2}{u}\,du \\ &= \ln 2 - \Big|_1^2 \big(\tfrac{1}{2}u^2 - 2u + \ln u\big) = \tfrac{1}{2}\end{aligned}$$

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(c) Let $u = 1 + x^{1/3}$. Then $x = (u-1)^3$, $dx = 3(u-1)^2\,du$, and

$$\int_0^{27} \frac{x^{1/3}}{1+x^{1/3}}\,dx = \int_1^4 \frac{u-1}{u} 3(u-1)^2\,du = \int_1^4 \frac{3u^3 - 9u^2 + 9u - 3}{u}\,du = \cdots = \frac{45}{2} - 3\ln 4$$

4.2

4.2.2 We use formula (4.2.1) and then introduce $u = \alpha x^2$ as a new variable, assuming $\alpha \neq 0$. This yields

$$F'(\alpha) = \int_0^1 \frac{\partial}{\partial\alpha}(xe^{\alpha x^2})\,dx = \int_0^1 x^3 e^{\alpha x^2}\,dx = \frac{1}{2\alpha^2}\int_0^\alpha ue^u\,du = \frac{1}{2\alpha^2}\Big|_0^\alpha (ue^u - e^u) = \frac{\alpha e^\alpha - e^\alpha + 1}{2\alpha^2}$$

Direct calculation of F yields $F(\alpha) = \dfrac{1}{2\alpha}\Big|_0^1 e^{\alpha x^2} = \dfrac{e^\alpha - 1}{2\alpha}$, which gives $F'(\alpha) = \dfrac{\alpha e^\alpha - e^\alpha + 1}{2\alpha^2}$, confirming the result above.

We assumed above that $\alpha \neq 0$. If $\alpha = 0$, then formula (4.2.1) yields $F'(0) = \int_0^1 x^3\,dx = 1/4$. To get the answer by differentiating F directly, we need to know that $F(0) = \int_0^1 x\,dx = 1/2$. Then

$$F'(0) = \lim_{\alpha\to 0} \frac{F(\alpha) - F(0)}{\alpha} = \lim_{\alpha\to 0}\frac{e^\alpha - 1 - \alpha}{2\alpha^2} = \text{“}\frac{0}{0}\text{”} = \cdots = \frac{1}{4}$$

as it should be.

4.2.6 By Leibniz's formula (Theorem 4.2.1),

$$\dot{x}(t) = e^{-\delta(t-\tau)}y(t) + \int_{-\infty}^t -\delta e^{-\delta(t-\tau)}y(\tau)\,d\tau = y(t) - \delta x(t)$$

4.2.8 See the answer in the book. In order to use Leibniz's formula (Theorem 4.2.2 in this case) we need to know that there exist functions $p(\tau)$ and $q(\tau)$ such that $\int_{-\infty}^t p(\tau)\,d\tau$ and $\int_{-\infty}^t q(\tau)\,d\tau$ converge, and such that $|f'(t-\tau)k(\tau)| \le p(\tau)$ and $|G'_t(\tau,t)| \le q(\tau)$ for all $\tau \le t$. Since $G'(\tau,t) = -k(\tau)f(t-\tau)$, the inequality for G'_t boils down to $|k(\tau)f(t-\tau)| \le q(\tau)$ for all $\tau \le t$.

4.2.10 (a) $g'(Q) = c + h\int_0^Q f(D)\,dD - p\int_Q^a f(D)\,dD$, and $g''(Q) = (h+p)f(Q) \ge 0$.

(b) Since $\int_0^a f(D)\,dD = 1$, we have

$$\int_Q^a f(D)\,dD = \int_0^a f(D)\,dD - \int_0^Q f(D)\,dD = 1 - \int_0^Q f(D)\,dD$$

and therefore

$$g'(Q) = c - p + (h+p)\int_0^Q f(D)\,dD$$

Since Q^* is the minimum point of $g(Q)$, it must be a stationary point—that is, $g'(Q^*) = 0$. Therefore $c - p + (h+p)F(Q^*) = 0$, which implies $F(Q^*) = (p-c)/(p+h)$.

4.3

4.3.2 From the functional equation (4.3.2), $\Gamma(\frac{3}{2}) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$. The given formula is thus correct for $n = 1$. Suppose it is correct for $n = k$. Then, using (4.3.2),

$$\begin{aligned}\Gamma(k+1+\tfrac{1}{2}) &= \Gamma((k+\tfrac{1}{2})+1) = (k+\tfrac{1}{2})\Gamma(k+\tfrac{1}{2}) = (k+\tfrac{1}{2})\frac{(2k-1)!}{2^{2k-1}(k-1)!}\sqrt{\pi} \\ &= \frac{2k+1}{2}\,\frac{(2k-1)!}{2^{2k-1}(k-1)!}\sqrt{\pi} = \frac{(2k-1)!\,2k(2k+1)}{2\cdot 2k\cdot 2^{2k-1}(k-1)!}\sqrt{\pi} = \frac{(2k+1)!}{2^{2k+1}k!}\sqrt{\pi}\end{aligned}$$

Thus the proposed formula is valid also for $n = k+1$. By mathematical induction it is true for all natural numbers n.

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4.3.5 (a) Introduce $u = \lambda x$ as a new variable. Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1}e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty (u/\lambda)^{\alpha-1}e^{-u}\frac{1}{\lambda}\,du = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1}e^{-u}\,du = 1$$

(b) $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \dfrac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1}e^{-(\lambda-t)x}\,dx$. With $u = (\lambda - t)x$ as a new variable we get

$$M(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{u^{\alpha-1}e^{-u}}{(\lambda-t)^\alpha}\,du = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \Big(\frac{\lambda}{\lambda-t}\Big)^\alpha = \lambda^\alpha(\lambda-t)^{-\alpha}$$

Differentiation gives $M'(t) = \alpha\lambda^\alpha(\lambda-t)^{-\alpha-1}$ and in general

$$M^{(n)}(t) = \alpha(\alpha+1)\cdots(\alpha+n-1)\lambda^\alpha(\lambda-t)^{-\alpha-n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{\lambda^\alpha}{(\lambda-t)^{\alpha+n}}$$

Hence,

$$M'(0) = \frac{\alpha}{\lambda} \qquad \text{and} \qquad M^{(n)}(0) = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\lambda^n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\lambda^n}$$

(Alternatively we could have used the formula $M^{(n)}(0) = \int_{-\infty}^{\infty} x^n f(x)\,dx$ from Problem 4.2.5.)

4.4

4.4.2 The inner integral is $\dfrac{1}{x^2}\displaystyle\int_0^b \frac{1}{x}e^{y/x}\,dy = \frac{1}{x^2}\Big|_{y=0}^{y=b} e^{y/x} = \frac{1}{x^2}e^{b/x} - \frac{1}{x^2}$, so $I = \displaystyle\int_1^a \Big(\frac{1}{x^2}e^{b/x} - \frac{1}{x^2}\Big)\,dx$. With $w = \dfrac{1}{x}$, we get $x = \dfrac{1}{w}$ and $dx = -\dfrac{1}{w^2}\,dw$. Therefore $I = \displaystyle\int_1^{1/a} (w^2e^{bw} - w^2)\Big(-\frac{1}{w^2}\Big)\,dw = \int_1^{1/a} (-e^{bw}+1)\,dw = \frac{1}{b}(e^b - e^{b/a}) + \frac{1}{a} - 1$.

4.4.3 The integral $I = \iint_R f(x,y)\,dx\,dy$ is

$$\begin{aligned}
I &= \int_0^1 \Big(\int_0^a \frac{2k}{(x+y+1)^3}\,dx\Big)\,dy = \int_0^1 \Big(\Big|_{x=0}^{x=a} -\frac{k}{(x+y+1)^2}\Big)\,dy \\
&= \int_0^1 \Big(\frac{k}{(y+1)^2} - \frac{k}{(y+a+1)^2}\Big)\,dy = \Big|_0^1 \Big(-\frac{k}{y+1}\ \frac{k}{y+a+1}\Big) \\
&= -\frac{k}{2} + \frac{k}{a+2} + k - \frac{k}{a+1} = \frac{k(a^2+3a)}{2(a^2+3a+2)}
\end{aligned}$$

The integral equals 1 if $k = k_a = \dfrac{2(a^2+3a+2)}{a^2+3a} = 2 + \dfrac{4}{a^2+3a}$. Obviously, $k_a > 2$ if $a > 0$.

4.4.5 The innermost integral is

$$\int_0^1 (x_1^2 + x_2^2 + \cdots + x_n^2)\,dx_1 = \Big|_0^1 [\tfrac{1}{3}x_1^3 + x_1(x_2^2 + x_3^2 + \cdots + x_n^2)] = \tfrac{1}{3} + x_2^2 + x_3^2 + \cdots + x_n^2$$

Next,

$$\int_0^1 (\tfrac{1}{3} + x_2^2 + x_3^2 + \cdots + x_n^2)\,dx_2 = \Big|_0^1 [\tfrac{1}{3}x_2 + \tfrac{1}{3}x_2^3 + x_2(x_3^2 + x_4^2 + \cdots + x_n^2)] = \tfrac{2}{3} + x_3^2 + x_4^2 + \cdots + x_n^2$$

etc. By induction, $I = n/3$.

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4.5

4.5.2 If $y \in [0, 1]$, then x runs from 0 to y, and if $y \in [1, 2]$, then x must run from $\sqrt{y-1}$ to 1. Thus,

$$V = \int_0^1 \Big(\int_0^y xy^2\,dx\Big)\,dy + \int_1^2 \Big(\int_{\sqrt{y-1}}^1 xy^2\,dx\Big)\,dy = \int_0^1 \frac{y^4}{2}\,dy + \int_1^2 \Big(y^2 - \frac{y^3}{2}\Big)\,dy = \frac{67}{120}$$

4.5.6 $|x-y| = x-y$ if $x \ge y$ and $y-x$ if $x < y$. Hence (see Fig. A4.5.6(a) in the book),

$$\int_0^1 \int_0^1 |x-y|\,dx\,dy = \iint_A |x-y|\,dx\,dy + \iint_B |x-y|\,dx\,dy$$
$$= \int_0^1 \Big[\int_0^y (y-x)\,dx + \int_y^1 (x-y)\,dx\Big]\,dy = \int_0^1 \Big[\Big|_{x=0}^{x=y}\big(yx - \frac{1}{2}x^2\big) + \Big|_{x=y}^{x=1}\big(\frac{1}{2}x^2 - yx\big)\Big]\,dy$$
$$= \int_0^1 \big[y^2 - \frac{1}{2}y^2 + \frac{1}{2} - y - \frac{1}{2}y^2 + y^2\big]\,dy = \Big|_0^1 \big(y^2 - y + \frac{1}{2}\big)\,dy = \Big|_0^1 \big(\frac{1}{3}y^3 - \frac{1}{2}y^2 + \frac{1}{2}y\big) = \frac{1}{3}$$

4.6

4.6.1 (a) Use the same subdivision as in Example 4.6.1, except that $j = 0, \dots, 2n-1$. Then

$$(2x_i^* - y_j^* + 1)\,\Delta x_i\,\Delta y_j = \Big(2\frac{i}{n} - \frac{j}{n} + 1\Big)\frac{1}{n}\frac{1}{n} = 2\frac{i}{n^3} - \frac{j}{n^3} + \frac{1}{n^2}$$

and

$$\sum_{j=0}^{2n-1}\sum_{i=0}^{n-1}\Big(2\frac{i}{n^3} - \frac{j}{n^3} + \frac{1}{n^2}\Big) = 2\frac{1}{n^3}\sum_{j=0}^{2n-1}\Big(\sum_{i=0}^{n-1} i\Big) - \frac{1}{n^3}\sum_{i=0}^{n-1}\Big(\sum_{j=0}^{2n-1} j\Big) + \frac{1}{n^2}\sum_{j=0}^{2n-1}\Big(\sum_{i=0}^{n-1} 1\Big)$$
$$= 2\frac{1}{n^3}\sum_{j=0}^{2n-1}\frac{1}{2}n(n-1) - \frac{1}{n^3}\sum_{i=0}^{n-1}\frac{1}{2}(2n-1)2n + \frac{1}{n^2}\sum_{j=0}^{2n-1} n$$
$$= 2\frac{1}{n^3}\frac{1}{2}n(n-1)2n - \frac{1}{n^3}\frac{1}{2}(2n-1)2nn + \frac{1}{n^2}n2n = 2 - \frac{1}{n} \to 2$$

as $n \to \infty$.

4.7

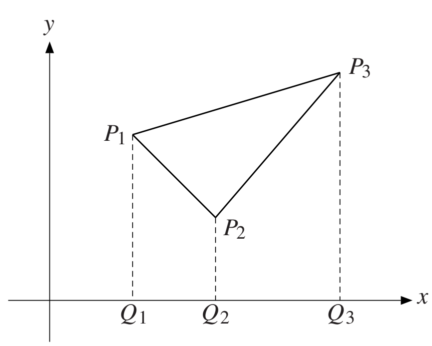

Figure M4.7.2

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4.7.2 Assume for convenience that the points $P_i = (x_i, y_i)$, $i = 1, 2, 3$, all lie in the first quadrant, and that $x_1 \le x_2 \le x_3$. Figure M4.7.2 shows the triangle $P_1P_2P_3$ together with the normals from the P_i to the corresponding points Q_i on the x-axis. For each pair (i, j) with $i < j$, the points P_i, Q_i, Q_j, and P_j form a quadrilateral with two parallel sides (called a trapezium in Britain, a trapezoid in the US), whose area is $T_{ij} = \frac{1}{2}(x_j - x_i)(y_i + y_j)$. If P_2 lies below the line P_1P_3, then the area of the triangle is $T_{13} - T_{12} - T_{23}$, and an easy computation shows that this equals $A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$. If P_2 lies above P_1P_3, then the area of the triangle is $T_{12} + T_{23} - T_{13} = -A$. In either case the area equals $|A|$.

If the x_i are in a different order from what we assumed above, we can renumber them. That may change the sign of the determinant but not its absolute value. Finally, if the triangle does not lie in the first quadrant, we can move it there by a parallel translation. Such a translation will not change the area of the triangle, nor will it change the value of the determinant, since we are just adding multiples of the first column to the other two columns.

4.7.3 (b) In this problem it is convenient to use polar coordinates centred at $(0, 1)$, so let $x = r\cos\theta$, $y = 1 + r\sin\theta$. The Jacobian is $\partial(x, y)/\partial(r, \theta) = r$ in this situation too, and

$$\iint_A x^2\,dx\,dy = \int_0^{2\pi}\Big(\int_0^{1/2} r^3\cos^2\theta\,dr\Big)\,d\theta = \int_0^{2\pi}\Big(\Big|_{r=0}^{r=1/2}\frac{r^4}{4}\cos^2\theta\Big)\,d\theta = \frac{1}{64}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{\pi}{64}$$

4.7.5 (b) Introduce new variables $u = y - 2x$, $v = 3x + y$. Then $x = -\frac{1}{5}u + \frac{1}{5}v$, $y = \frac{3}{5}u + \frac{2}{5}v$ and $J = \dfrac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{5}$. It follows that

$$\iint_{A_2}(x + y)\,dx\,dy = \int_4^8\Big(\int_{-1}^1\Big(\frac{2}{5}u + \frac{3}{5}v\Big)|J|\,du\Big)\,dv = \int_4^8\Big(\int_{-1}^1\Big(\frac{2}{5}u + \frac{3}{5}v\Big)\frac{1}{5}\,du\Big)\,dv = 144/25$$

4.8

4.8.1 (a) Use polar coordinates and let $A_n = \{(x, y) : 1 \le x^2 + y^2 \le n^2\}$. Then

$$\iint_{A_n}(x^2+y^2)^{-3}\,dx\,dy = \int_0^{2\pi}\Big(\int_1^n r^{-6}r\,dr\Big)\,d\theta \overset{*}{=} 2\pi\int_1^n r^{-5}\,dr = \tfrac{1}{2}\pi(1-n^{-4}) \to \tfrac{1}{2}\pi \quad \text{as } n \to \infty$$

About the equality $\overset{*}{=}$: The integral $J_n = \int_1^n r^{-6}r\,dr$ is independent of θ, therefore $\int_0^{2\pi} J_n\,d\theta = 2\pi J_n$.

(b) With polar coordinates and with A_n as in part (a), we get

$$I_n = \iint_{A_n}(x^2 + y^2)^{-p}\,dx\,dy = 2\pi\int_1^n r^{1-2p}\,dr = \begin{cases} 2\pi\dfrac{n^{2-2p} - 1}{2 - 2p} & \text{if } p \ne 1 \\ 2\pi\ln n & \text{if } p = 1 \end{cases}$$

If $p > 1$, then $I_n \to \pi/(p - 1)$ as $n \to \infty$, but if $p \le 1$, then $I_n \to \infty$.

4.8.4 Note that the integrand takes both positive and negative values in the domain of integration. In the calculations of the two iterated integrals the positive and the negative parts will more or less balance

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each other, but not in exactly the same way. The two iterated integrals $\int_1^d \int_1^b y(y+x)^{-3}\,dx\,dy$ and $\int_1^b \int_1^d y(y+x)^{-3}\,dy\,dx$ both tend to ∞ as b and d tend to ∞, and so do the integrals of $x(y+x)^{-3}$. The trouble arises when we try to take the difference. That leads us into an $\infty - \infty$ situation that does not lead to any definite value.

4.8.6 (b) Introduce new variables $u = x + y$, $v = x - y$. Then $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$, and the Jacobian determinant is $\partial(x, y)/\partial(u, v) = -1/2$. The square $B'_n = [-n, n] \times [-n, n]$ in the uv-plane corresponds to the square B_n with corners $(2n, 0)$, $(0, 2n)$, $(-2n, 0)$, and $(0, -2n)$ in the xy-plane, and $I_n = \iint_{B_n} \frac{e^{-(x-y)^2}}{1+(x+y)^2}\,dx\,dy = \int_{-n}^{n}\Big(\int_{-n}^{n} \frac{e^{-v^2}}{1+u^2}\frac{1}{2}\,du\Big)\,dv = \int_{-n}^{n} e^{-v^2}\,dv \int_{-n}^{n} \frac{1}{2}\frac{1}{1+u^2}\,du = \int_{-n}^{n} e^{-v^2}\,dv \cdot \arctan n$. Let $n \to \infty$. Then $I_n \to \int_{-\infty}^{\infty} e^{-v^2}\,dv \cdot \frac{\pi}{2} = \sqrt{\pi}\cdot\frac{\pi}{2} = \frac{\pi^{3/2}}{2}$. (Here we used Poisson's integral formula (4.3.3).)

4.8.7 (b) With polar coordinates and with the sets A_n as in Example 3,

$$\iint_A \frac{-\ln(x^2+y^2)}{\sqrt{x^2+y^2}}\,dx\,dy = \lim_{n\to\infty}\int_0^{\pi/2}\Big(\int_{1/n}^{1} -\frac{\ln r^2}{r}\,r\,dr\Big)\,d\theta = -\pi \lim_{n\to\infty}\int_{1/n}^{1} \ln r\,dr = \cdots = \pi$$

Chapter 5 Differential Equations I: First-order Equations in One Variable

5.1

5.1.6 The statement of the problem is slightly inaccurate. Instead of "for all t" it should have said "for all t in some open interval I around 0". With that modification the answer in the book is quite correct. (Actually, the given differential equation has no solution defined on the entire real line. One can show that, with the initial condition $x(0) = 0$, the equation has the solution $x = \tan(\frac{1}{2}t^2)$ over $(-\sqrt{\pi}, \sqrt{\pi})$, but this solution cannot be extended to any larger interval because $x(t)$ runs off to infinity as t approaches either endpoint.

5.2

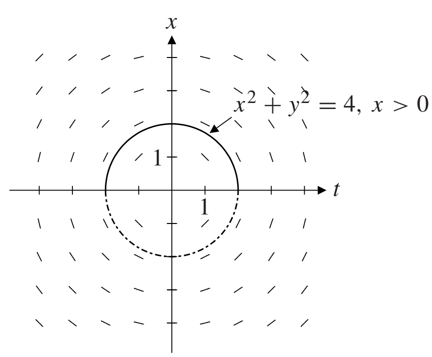

Figure M5.2.2

5.2.2 The solution curves are semicircles (not full circles) of the form $t^2 + x^2 = C$, $x \neq 0$, with C an arbitrary positive constant. (This can be shown by direct differentiation, or by solving the separable

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equation $\dot{x} = -t/x$ using the method set out in the next section.) The integral curve through $(0, 2)$ is given by $t^2 + x^2 = 4$, $x > 0$, in other words it is the graph of the function $x(t) = \sqrt{4 - t^2}$ over the interval $(-2, 2)$. The lower semicircle shown in Fig. M5.2.2 is the graph of another function, $x(t) = -\sqrt{4 - t^2}$, which is also a solution of the differential equation, but not the one you were asked to find. (The figure in answer in the book is misleading since it shows a full circle.)

5.3

5.3.3 (a) One constant solution, $x \equiv 0$. Otherwise, separating the variables,

$$\int \frac{dx}{x} = \int \frac{1-t}{t}\,dt = \int \Big(\frac{1}{t} - 1\Big)\,dt \quad \Longrightarrow \quad \ln|x| = \ln|t| - t + C_1$$

Hence, $|x| = e^{\ln|t| - t + C_1} = e^{\ln|t|}e^{-t}e^{C_1} = C_2|t|e^{-t}$, so $x = Cte^{-t}$, where $C = \pm C_2 = \pm e^{C_1}$. The integral curve through $(t_0, x_0) = (1, 1/e)$ is $x = te^{-t}$.

(b) One constant solution, $x \equiv 0$. Otherwise,

$$\frac{\dot{x}}{x} = \frac{t^2}{1+t^3} \quad \Longrightarrow \quad \ln|x| = \tfrac{1}{3}\ln|1+t^3| + C_1 \quad \Longrightarrow \quad x = C\sqrt[3]{1+t^3}$$

Integral curve through $(t_0, x_0) = (0, 2)$ for $C = 2$.

(c) No constant solutions. $\displaystyle\int x\,dx = \int t\,dt \Longrightarrow \tfrac{1}{2}x^2 = \tfrac{1}{2}t^2 + C_1 \Longrightarrow x^2 = t^2 + C$, where $C = 2C_1$. An integral curve through $(t_0, x_0) = (\sqrt{2}, 1)$ must have $C = -1$ and $x > 0$, so $x = \sqrt{t^2 - 1}$.

(d) The equation is equivalent to $\dot{x} = e^{-2t}(x+1)^2$. One constant solution, $x \equiv -1$. The nonconstant solutions are found by separating the variables:

$$\int \frac{dx}{(x+1)^2} = \int e^{-2t}\,dt \quad \Longrightarrow \quad -\frac{1}{x+1} = -\tfrac{1}{2}e^{-2t} + C_1 \quad \Longrightarrow \quad x = \frac{1 - e^{-2t} + C}{1 + e^{-2t} - C}$$

where $C = 1 + 2C_1$. For the solution to pass through (t_0, x_0) we must have $1 - e^0 + C = 0$, i.e. $C = 0$, so $x = \dfrac{1 - e^{-2t}}{1 + e^{-2t}}$.

5.3.6 For convenience we shall suppose that $x > 0$ and $t > 0$.

(a) The equation $\dfrac{t\dot{x}}{x} = a$ is separable, and we get

$$\int \frac{dx}{x} = \int \frac{a}{t}\,dt \quad \Longrightarrow \quad \ln x = a\ln t + C_1 = \ln t^a + C_1 \quad \Longrightarrow \quad x = e^{\ln x} = t^a \cdot e^{C_1} = Ct^a$$

where $C = e^{C_1}$.

(b) Here, $\dfrac{t\dot{x}}{x} = at + b$. That gives

$$\int \frac{dx}{x} = \int \frac{at+b}{t}\,dt = \int \Big(a + \frac{b}{t}\Big)\,dt \quad \Longrightarrow \quad \ln x = at + b\ln t + C_1 \quad \Longrightarrow \quad x = Ce^{at}t^b$$

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(c) The equation $\dfrac{t\dot{x}}{x} = ax + b$ gives $\dfrac{\dot{x}}{x(ax+b)} = \dfrac{1}{t}$. Since $\dfrac{1}{x(ax+b)} = \dfrac{1}{b}\Big(\dfrac{1}{x} - \dfrac{a}{ax+b}\Big)$, we get

$$\int \Big(\frac{1}{x} - \frac{a}{ax+b}\Big)\,dx = \int \frac{b}{t}\,dt \implies \ln x - \ln|ax+b| = b\ln t + C_1 \implies \frac{x}{ax+b} = Ct^b$$

where $C = \pm e^{C_1}$, and finally $x = \dfrac{Cbt^b}{1 - Cat^b}$.

5.3.8 (a) Let $P = An_0^\alpha a^b$ and $Q = \alpha v + \varepsilon$. Separating the variables we get $\int K^{c-b}\,dK = P\int e^{Qt}\,dt$. Integration yields $\dfrac{K^{1-b+c}}{1-b+c} = \dfrac{P}{Q}e^{Qt} + C_1$. Hence, $K = \Big[\dfrac{P}{Q}(1-b+c)e^{Qt} + C\Big]^{1/(1-b+c)}$, where $C = C_1(1-b+c)$.

(b) We separate the variables and get $\displaystyle\int \frac{x\,dx}{(\beta - \alpha x)(x-a)} = \int dt$. The hint in the problem gives

$$\int \frac{\beta\,dx}{\beta - \alpha x} + \int \frac{a\,dx}{x-a} = (\beta - \alpha a)\int dt \iff -\frac{\beta}{\alpha}\ln|\beta - \alpha x| + a\ln|x-a| = (\beta - \alpha a)t + C_1$$

$$\iff \frac{\beta}{\alpha}\ln|\beta - \alpha x| - a\ln|x-a| = \ln|\beta - \alpha x|^{\beta/\alpha} + \ln|x-a|^{-a} = -(\beta - \alpha a)t - C_1$$

$$\iff |\beta - \alpha x|^{\beta/\alpha}|x-a|^{-a} = e^{-(\beta-\alpha a)t - C_1} = e^{-(\beta - \alpha a)t}e^{-C_1} = Ce^{(\alpha a - \beta)t}$$

where $C = e^{-C_1}$. We have the same answer as in the book (since $|\beta - \alpha x| = |\alpha x - \beta|$).

5.3.9 (a) Note first that $L = L_0 e^{\lambda t} = [L_0^\alpha e^{\alpha\lambda t}]^{1/\alpha}$, so as $t \to \infty$,

$$\frac{K}{L} = \frac{\big[K_0^\alpha + (sA/\lambda)L_0^\alpha(e^{\alpha\lambda t} - 1)\big]^{1/\alpha}}{[L_0^\alpha e^{\alpha\lambda t}]^{1/\alpha}} = \Big[\frac{K_0^\alpha}{L_0^\alpha e^{\alpha\lambda t}} + \frac{sA}{\lambda}(1 - e^{-\alpha\lambda t})\Big]^{1/\alpha} \to \Big(\frac{sA}{\lambda}\Big)^{1/\alpha}$$

and

$$\frac{X}{L} = \frac{AK^{1-\alpha}L^\alpha}{L} = A\Big(\frac{K}{L}\Big)^{1-\alpha} \to A\Big(\frac{sA}{\lambda}\Big)^{(1-\alpha)/\alpha} = A^{1/\alpha}\Big(\frac{s}{\lambda}\Big)^{(1-\alpha)/\alpha}$$

(b) The equation is separable, $dK/dt = sAK^{1-\alpha}L^\alpha = sAK^{1-\alpha}b^\alpha(t+a)^\alpha$. We get

$$\int K^{\alpha-1}\,dK = sAb^\alpha \int (t+a)^{p\alpha}\,dt \implies \frac{1}{\alpha}K^\alpha = \frac{sAb^\alpha}{p\alpha + 1}(t+a)^{p\alpha+1} + C$$

The initial condition gives $C = \dfrac{1}{\alpha}K_0^\alpha - \dfrac{sAb^\alpha}{p\alpha+1}a^{p\alpha+1}$, and so

$$K = \big[K_0^\alpha + s\alpha Ab^\alpha\big((t+a)^{p\alpha+1} - a^{p\alpha+1}\big)/(p\alpha+1)\big]^{1/\alpha}$$

It follows that

$$\frac{K}{L} = \frac{\Big[K_0^\alpha + \dfrac{s\alpha Ab^\alpha}{p\alpha+1}\big((t+a)^{p\alpha+1} - a^{p\alpha+1}\big)\Big]^{1/\alpha}}{\big[b^\alpha(t+\alpha)^{p\alpha}\big]^{1/\alpha}}$$

$$= \Big[\frac{K_0^\alpha}{b^\alpha(t+\alpha)^{p\alpha}} + \frac{s\alpha A}{p\alpha+1}\Big(t + a - \frac{a^{p\alpha+1}}{(t+\alpha)^{p\alpha}}\Big)\Big]^{1/\alpha} \to \infty \quad \text{as} \quad t \to \infty$$

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5.4

5.4.6 Use formula (5.4.6) to solve these equations.

(b) Here $\int a(t)\,dt = -\int (1/t)\,dt = -\ln t$, and (5.4.6) yields the solution $x = Ct + t^2$.

(c) In this case, $\int a(t)\,dt = -\frac{1}{2}\ln(t^2-1)$, and (5.4.6) yields the solution $x = C\sqrt{t^2-1} + t^2 - 1$.

(d) Here $a(t) = -2/t$, $b(t) = -2a^2/t^2$, $\int a(t)\,dt = -2\ln t$, which leads to $x = Ct^2 + 2a^2/3t$.

5.4.9 From $x = X/N$, by logarithmic differentiation, $\dot{x}/x = \dot{X}/X - \dot{N}/N$. Moreover, (ii) implies that $\dot{X}/X = a\dot{N}/N$, so $\dot{x}/x = (a-1)\dot{N}/N = (a-1)[\alpha - \beta(1/x)]$. It follows that $\dot{x} = (a-1)\alpha x - (a-1)\beta$. The solution is $x(t) = [x(0) - \beta/\alpha]e^{\alpha(a-1)t} + \beta/\alpha$. Then (ii) and $x = X/N$ together imply that $N(t) = [x(t)/A]^{1/(a-1)}$, $X(t) = A[N(t)]^a$. If $0 < a < 1$, then $x(t) \to \beta/\alpha$, $N(t) \to (\beta/\alpha A)^{1/(a-1)}$, and $X(t) \to A(\beta/\alpha A)^{a/(a-1)}$ as $t \to \infty$.

5.4.10 (b) It suffices to note that $(1 - e^{-\xi t})/\xi > 0$ whenever $\xi \neq 0$ (look at $\xi > 0$ and $\xi < 0$ separately). Then apply this with $\xi = \alpha\sigma - \mu$. Faster growth per capita is to be expected because foreign aid contributes positively.

(c) Using equation (∗∗), we get $x(t) = \left[x(0) + \left(\frac{\sigma}{\alpha\sigma - \mu}\right)\frac{H_0}{N_0}\right]e^{-(\rho-\alpha\sigma)t} + \left(\frac{\sigma}{\mu - \alpha\sigma}\right)\frac{H_0}{N_0}e^{(\mu-\rho)t}$. Note that, even if $\alpha\sigma < \rho$, $x(t)$ is positive and increasing for large t as long as $\mu > \rho$. So foreign aid must grow faster than the population.

5.5

5.5.2 With $f(t,x) = 1$ and $g(t,x) = t/x + 2$, we have $(g_t' - f_x')/f = 1/x$, so we are in Case (II). It follows from (5.5.11) that we can choose $\beta(x) = \exp(\int (1/x)\,dx) = \exp(\ln x) = x$ as an integrating factor. Hence, $x + (t+2x)\dot{x} = 0$ is exact, and (8) easily yields $h(t,x) = tx + x^2 - t_0x_0 - x_0^2$. The solution of the differential equation is obtained from $tx + x^2 = C$, where C is a constant. We assumed that $t > 0$ and $x > 0$, so C will be positive, and the solution of $tx + x^2 = C$ is $x = -\frac{1}{2}t + \sqrt{\frac{1}{4}t^2 + C}$.

5.6

5.6.1 (a) With $t > 0$, the given equation is equivalent to the Bernoulli equation $\dot{x} + (2/t)x = x^r$ with $r = 2$. Let $z = x^{1-r} = x^{-1} = 1/x$, so that $x = 1/z$. Then $\dot{x} = -z^{-2}\dot{z}$ and the differential equation becomes

$$-z^{-2}\dot{z} + (2/t)z^{-1} = z^{-2} \iff \dot{z} - (2/t)z = -1$$

whose solution is $z = Ct^2 + t$. Thus $x = (Ct^2 + t)^{-1}$.

(b) The equation is a Bernoulli equation as in (5.6.1), with $r = 1/2$. Thus we substitute $z = x^{1-1/2} = x^{1/2}$, i.e. $x = z^2$. Then $\dot{x} = 2z\dot{z}$ and the given equation becomes

$$2z\dot{z} = 4z^2 + 2e^t z \iff \dot{z} - 2z = e^t$$

(Recall that x is assumed to be positive, and therefore $z > 0$.) Formula (5.4.4) yields

$$z = e^{2t}(C + \int e^{-2t}e^t\,dt) = e^{2t}(C + \int e^{-t}\,dt) = e^{2t}(C - e^{-t}) = Ce^{2t} - e^t.$$

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(Alternatively we could go back to the method that was used to deduce (5.4.4) and calculate like this:

$$\dot{z} - 2z = e^t \iff (\dot{z} - 2z)e^{-2t} = e^{-t}$$
$$\iff \frac{d}{dt}(ze^{-2t}) = e^{-t} \iff ze^{-2t} = -e^{-t} + C, \quad \text{etc.)}$$

The solution of the given equation is therefore

$$x = z^2 = (Ce^{2t} - e^t)^2$$

(c) As in part (a), we substitute $x = 1/z$, with $\dot{x} = -z^{-2}\,dz$. This leads to the differential equation

$$\dot{z} - (1/t)z = -(\ln t)/t$$

Formula (5.4.6) yields the solution

$$z = e^{\ln t}\Big(C - \int e^{-\ln t}\frac{\ln t}{t}\,dt\Big) = t\Big(C - \int \frac{\ln t}{t^2}\,dt\Big) = Ct + \ln t + 1 \implies x = (Ct + \ln t + 1)^{-1}$$

(The answer in the book also lists $x \equiv 0$ as a solution, and it certainly satisfies the equation, but the problem explicitly calls for solutions with $x > 0$.)

5.6.3 Introducing $z = K^{1-b}$ as a new variable, we find that (see (5.6.3)) $\dot{z} + Pz = Qe^{(av+\varepsilon)t}$, where $P = \alpha\delta(1-b)$ and $Q = \alpha A n_0^a(1-b)$. According to (5.4.4), the solution of this linear differential equation is

$$z = Ce^{-Pt} + Qe^{-Pt}\int e^{Pt}e^{(av+\varepsilon)t}\,dt = Ce^{-Pt} + Qe^{-Pt}\int e^{(av+\varepsilon+P)t}\,dt$$
$$= Ce^{-Pt} + Qe^{-Pt}\frac{1}{av+\varepsilon+P}e^{(av+\varepsilon+P)t} = Ce^{-Pt} + \frac{Qe^{(av+\varepsilon)t}}{av+\varepsilon+P}$$

Insert the values of P and Q. Then $K = z^{1/(1-b)}$ gives the answer in the book.

5.6.4 Introduce $z = K^{1-\alpha}$ as a new variable. By (5.6.3), we get the equation $\dot{z} - \gamma_2(1-\alpha)z = \gamma_1 b(1-\alpha)$. According to (5.4.3), the solution is $z = Ce^{\gamma_2(1-\alpha)t} - \gamma_1 b/\gamma_2$. Then $K = z^{1/(1-\alpha)}$ gives the answer in the book.

5.6.7 The equation is of the form $\dot{x} = g(x/t)$ with $g(z) = 1 + z - z^2$. According to Problem 5.6.6, the substitution $z = x/t$ leads to the separable equation $t\dot{z} = g(z) - z = 1 - z^2$. This has the two constant solutions $z = -1$ and $z = 1$. To find the other solutions, we separate the variables and get $\int \frac{dz}{1-z^2} = \int \frac{dt}{t}$. By a well-known identity, $\int \frac{1}{2}\Big(\frac{1}{1+z} + \frac{1}{1-z}\Big)dz = \int \frac{dt}{t} + C$. Integration yields $\frac{1}{2}\ln|1+z| - \frac{1}{2}\ln|1-z| = \ln|t| + C$, so

$$\ln\left|\frac{1+z}{1-z}\right| = 2\ln|t| + 2C = \ln t^2 + 2C \quad\implies\quad \frac{1+z}{1-z} = At^2$$

where $A = \pm e^{2C}$. Solving for z gives $z = \dfrac{At^2-1}{At^2+1}$, and finally $x = tz = \dfrac{At^3 - t}{At^2+1}$. In addition we have the two solutions $x = -t$ and $x = t$, corresponding to $z = \pm 1$.

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5.7

5.7.3 (a) This is a separable equation with solution $x(t) = (1+Ae^t)/(1-Ae^t)$ where $A = (x_0-1)/(x_0+1)$ for $x_0 \neq -1$. For $x_0 = -1$ we get the constant solution $x(t) \equiv -1$. For $x_0 \neq 1$, $x(t) \to -1$ as $t \to \infty$. If $x_0 > 1$, which occurs when $0 < A < 1$, then $x(t) \to \infty$ as $t \to (-\ln A)^-$, and $x(t) \to -\infty$ as $t \to (-\ln A)^+$. See Fig. A5.7.3(a) in the answer section of the book for some integral curves.

5.7.4 (a) $\partial k^*/\partial s = f(k^*)/[\lambda - sf'(k^*)] > 0$ and $\partial k^*/\partial \lambda = -k^*/[\lambda - sf'(k^*)] < 0$ when $\lambda > sf'(k^*)$. In equilibrium, capital per worker increases as the savings rate increases, and decreases as the growth rate of the work force increases. From $F(K, L) = Lf(k)$ with $k = K/L$, we obtain $F'_K(K, L) = Lf'(k)(1/L) = f'(k)$.

(b) From equations (i) to (iv), $c = (X - \dot{K})/L = (1 - s)X/L = (1 - s)f(k)$. But $sf(k^*) = \lambda k^*$, so when $k = k^*$ we have $c = f(k^*) - \lambda k^*$. The necessary first-order condition for this to be maximized w.r.t. k^* is that $f'(k^*) = \lambda$. But $F(K, L) = Lf(k)$ and so $F'_K = Lf'(k)dk/dK = f'(k)$ because $k = K/L$ with L fixed. Thus $\partial F/\partial K = \lambda$.

(c) See the answer in the book.

5.8

5.8.4 The equation is separable, and the solution through $(t_0, x_0) = (0, \frac{1}{2})$ is $x = 1/(1 + e^{-t})$. If condition (3) in Theorem 5.8.3 were satisfied, then for every t, there would exist numbers $a(t)$ and $b(t)$ such that $|x(1 - x)| \leq a(t)|x| + b(t)$ for all x. But then $|1 - x| \leq a(t) + b(t)/|x|$, which clearly is impossible when x is sufficiently large. Similarly, (4) implies $x^2(1 - x) \leq a(t)x^2 + b(t)$, so $1 - x \leq a(t) + b(t)/x^2$, which becomes impossible as $x \to -\infty$.

5.8.5 See Fig. M5.8.5. For $t < a$, $\dot{\varphi}(t) = -2(t - a) = 2(a - t) = 2\sqrt{(a - t)^2} = 2\sqrt{|\varphi(t)|}$. The argument for $t > b$ is similar. For t in (a, b) we have $\dot{\varphi}(t) = 0 = 2\sqrt{|\varphi(t)|}$. For $t < a$, $(\varphi(t) - \varphi(a))/(t - a) = -(t - a)^2/(t - a) = -(t - a) = a - t$, and for t slightly larger than a, $(\varphi(t) - \varphi(a))/(t - a) = 0$. It follows that when t is near a, $\left|(\varphi(t) - \varphi(a))/(t - a)\right| \leq |t - a|$, so φ is differentiable at a, and $\dot{\varphi}(a) = \lim_{t \to a}(\varphi(t) - \varphi(a))/(t - a) = 0 = 2\sqrt{|\varphi(a)|}$. In the same way we show that the differential equation is satisfied at $t = b$.

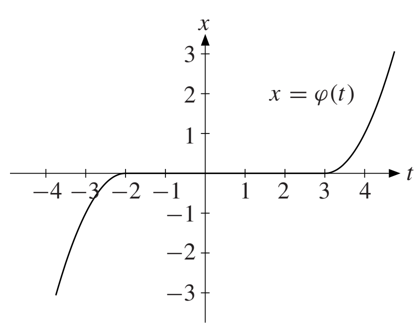

Figure M5.8.5

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6 Differential Equations II: Second-Order Equations and Systems in the Plane

6.1

6.1.4 In each of these three equations, let $u = \dot{x}$. That will give simple first-order equations for u which you can solve, and afterwards find x as $x = \int u\,dt$.

(a) Putting $u = \dot{x}$, we get $\dot{u} + 2u = 8$, which has the solution (see (5.4.3)) $u = Ce^{-2t} + 4$. But then $x = \displaystyle\int (Ce^{-2t} + 4)\,dt = -\tfrac{1}{2}Ce^{-2t} + 4t + B = Ae^{-2t} + 4t + B$, with $A = -\tfrac{1}{2}C$.

(b) With $u = \dot{x}$, we get $\dot{u} - 2u = 2e^{2t}$, with the solution (see (5.4.4))

$$u = Ce^{2t} + e^{2t}\int e^{-2t}2e^{2t}\,dt = Ce^{2t} + e^{2t}\int 2\,dt = Ce^{2t} + e^{2t}2t = (C + 2t)e^{2t}$$

Integration by parts then yields

$$\begin{aligned} x &= \int (C + 2t)e^{2t}\,dt = \tfrac{1}{2}(C + 2t)e^{2t} - \tfrac{1}{2}\int 2e^{2t}\,dt \\ &= \tfrac{1}{2}(C + 2t)e^{2t} - \tfrac{1}{2}e^{2t} + B = \tfrac{1}{2}(C - 1)e^{2t} + te^{2t} + B = Ae^{2t} + te^{2t} + B \end{aligned}$$

with $A = \tfrac{1}{2}(C - 1)$.

(c) Let $u = \dot{x}$. Then $\dot{u} - u = t^2$, which has the solution (see (5.4.4)) $u = Ae^t + e^t \displaystyle\int e^{-t}t^2\,dt$. Using integration by parts twice gives $\displaystyle\int e^{-t}t^2\,dt = -t^2e^{-t} - 2te^{-t} - 2e^{-t}$, and so $u = Ae^t - t^2 - 2t - 2$. Then $x = \displaystyle\int (Ae^t - t^2 - 2t - 2)\,dt = Ae^t - \tfrac{1}{3}t^3 - t^2 - 2t + B$.

6.1.6 (a) Suppose $x = \varphi(t)$ is a solution of $\ddot{x} = F(x, \dot{s})$. We know that if $\dot{x} = \dot{\varphi}(t) \neq 0$, then the equation $x - \varphi(t) = 0$ defines t as a function of x, with

$$\frac{dt}{dx} = -\frac{(\partial/\partial x)(x - \varphi(t))}{(\partial/\partial t)(x - \varphi(t))} = -\frac{\frac{\partial}{\partial x}(x - \varphi(t))}{\frac{\partial}{\partial t}(x - \varphi(t))} = \frac{1}{\dot{\varphi}(t)} = \frac{1}{\dot{x}}$$

Therefore $\dot{x} = 1/t'$, where $t' = dt/dx$. (The prime $'$ denotes differentiation with respect to x.) Now the chain rule gives

$$\ddot{x} = \frac{d}{dt}(\dot{x}) = \frac{d}{dt}\Big(\frac{1}{t'}\Big) = \frac{d}{dx}\Big(\frac{1}{t'}\Big)\frac{dx}{dt} = -\frac{t''}{(t')^2}\,\frac{1}{t'} = -\frac{t''}{(t')^3}$$

and the differential equation $\ddot{x} = F(x, \dot{x})$ becomes

$$-\frac{t''}{(t')^3} = F(x, 1/t') \quad\iff\quad t'' = -(t')^3F(x, 1/t')$$

(b) (i) Obviously, $x(t) \equiv C$ is a solution for any constant C. The equation for t'' in (a) becomes $t'' = -(t')^3(-x)(1/t'^3) = x$, or $d^2t/dx^2 = x$. Integration yields $dt/dx = \tfrac{1}{2}x^2 + A$, and further

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integration results in $t = \frac{1}{6}x^3 + Ax + B$, where A and B are arbitrary constants. A nonconstant solution of the equation $\ddot{x} = -x\dot{x}^3$ is therefore given implicitly by the equation $x^3 + A_1x + B_1 = 6t$.

(ii) In this case, $x(t) \equiv C$ is a solution for every constant $C \neq 0$. For solutions with $\dot{x} \neq 0$ we use the transformation in (a) and get

$$t'' = -(t')^3\Big(\frac{1/(t')^2}{x}\Big) = -\frac{t'}{x} \iff \frac{t''}{t'} = -\frac{1}{x} \iff \ln|t'| = -\ln|x| + A_1$$

This yields $t' = A_2/x$, where $A_2 = \pm e^{A_1}$, and then $t = \int t'\,dx = A_2 \ln|x| + B_1$. This yields $\ln|x| = At + B_2$, where $A = 1/A_2$ and $B_2 = -B_1/A_2$. Finally, $x = \pm e^{B_2}e^{At} = Be^{At}$.

(Note that the constants A and B here are different from 0. If we let $A = 0$ in Be^{at}, we recover the constant solutions mentioned at the beginning.)

6.2

6.2.3 (a) Direct calculations show that $\ddot{u}_1 + \dot{u}_1 - 6u_1 = 0$ and $\ddot{u}_2 + \dot{u}_2 - 6u_2 = 0$. Since u_1 and u_2 are not proportional, Theorem 6.2.1(a) says that the general solution of $\ddot{x} + \dot{x} - 6x = 0$ is $x = Au_1 + Bu_2 = Ae^{2t} + Be^{-3t}$.

(b) Theorem 6.2.1(b) now tells us that the general solution is $Au_1 + Bu_2 + u^*$, where u^* is any particular solution of the equation. Since the right-hand side of the equation is a polynomial of degree 1, we try to find a particular solution of the form $u^* = Ct + D$. We get $\ddot{u}^* + \dot{u}^* - 6u^* = C - 6(Ct + D) = -6Ct + (C - 6D)$, so u^* is a solution if and only if $-6C = 6$ and $C = 6D$, in other words if and only if $C = -1$ and $D = -1/6$. Thus the general solution of the given equation is $x = Ae^{2t} + Be^{-3t} - t - 1/6$.

6.2.5 Let $x = (t + k)^{-1}$. Then $\dot{x} = -(t + k)^{-2}$, $\ddot{x} = 2(t + k)^{-3}$, and

$$\begin{aligned}(t + a)(t + b)\ddot{x} &+ 2(2t + a + b)\dot{x} + 2x \\ &= (t + k)^{-3}\big[2(t + 2)(t + b) - 2(2t + a + b)(t + k) + 2(t + k)^2\big] \\ &= (t + k)^{-3}2\big[k^2 - (a + b)k + ab\big] = (t + k)^{-3}2(k - a)(k - b)\end{aligned}$$

Thus $x = (t + k)^{-1}$ solves the given differential equation if and only if $k = a$ or $k = b$. Since $a \neq b$, the functions $u_1 = (t + a)^{-1}$ and $u_2 = (t + b)^{-1}$ are not proportional, and the general solution of the given equation is $x = Au_1 + Bu_2 = a(t + a)^{-1} + B(t + b)^{-1}$.

6.3

6.3.2 (a) The characteristic equation is $r^2 - 1 = 0$, with roots $r = 1$ and $r = -1$. The general solution of the corresponding homogeneous differential equation is therefore (Case (I) in Theorem 6.3.1), $x = Ae^t + Be^{-t}$. To find a particular solution of the given equation we try $u^*(t) = p\sin t + q\cos t$. Then $\dot{u}^* = p\cos t - q\sin t$, and $\ddot{u}^* = -p\sin t + q\cos t$. Inserting this into the given equation yields $-2p\sin t - 2q\cos t = \sin t$. Thus $p = -1/2$ and $q = 0$. The general solution of $\ddot{x} - x = \sin t$ is therefore $x = Ae^t + Be^{-t} - \frac{1}{2}\sin t$.

(b) The general solution of the homogeneous equation is again $x = Ae^t + Be^{-t}$. Since e^{-t} is a solution of the homogeneous equation, we try $u^*(t) = pte^{-t}$. Then $\dot{u}^*(t) = pe^{-t} - pte^{-t}$ and $\ddot{u}^*(t) = -pe^{-t} - pe^{-t} + pte^{-t}$, which inserted into the given equation yields $-2pe^{-t} = e^{-t}$, so $p = -1/2$, and the general solution is $x = Ae^t + Be^{-t} - \frac{1}{2}te^{-t}$.

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(c) The characteristic equation is $r^2 - 10r + 25 = (r - 5)^2 = 0$, so $r = 5$ is a double root. The general solution of the homogeneous equation is therefore (Case (II) in Theorem 6.3.1), $x = Ae^{5t} + Bte^{5t}$. To find a particular solution we try $u^*(t) = pt + q$, and find $p = 2/75$, $q = 3/125$. Hence, the general solution of the given equation is $x = Ae^{5t} + Bte^{5t} + \frac{2}{75}t + \frac{3}{125}$.

6.3.3 (a) The characteristic equation is $r^2+2r+1 = (r+1)^2 = 0$, so $r = -1$ is a double root. The general solution of the homogeneous equation is therefore (Case (II) in Theorem 6.3.1), $x = Ae^{-t} + Bte^{-t}$. To find a particular solution we try $u^*(t) = Ct^2 + Dt + E$. Then $\dot{u}^* = 2Ct + D$, and $\ddot{u}^* = 2C$. Inserted into the given equation this yields $2C+4Ct+2D+Ct^2+Dt+E = t^2$, or $Ct^2+(4C+D)t+2C+2D+E = t^2$. Equating like powers of t yields $C = 1$, $4C + D = 0$, and $2C + 2D + E = 0$. It follows that $D = -4$, and $E = 6$. So the general solution is $x = Ae^{-t} + Bte^{-t} + t^2 - 4t + 6$. The constants A and B are determined by the initial conditions. The condition $x(0) = 0$ yields $A + 6 = 0$, so $A = -6$. Since $\dot{x}(t) = -Ae^{-t} + Be^{-t} - Bte^{-t} + 2t - 4$, the condition $\dot{x}(0) = 1$ implies $-A + B - 4 = 1$, and so $B = A + 5 = -1$. The required solution is $x = -6e^t - te^t + t^2 - 4t + 6$.

(b) The characteristic equation is $r^2 + 4 = 0$, so $r = \pm 2i$ are the complex roots. The general solution of the homogeneous equation is therefore (Case (III) in Theorem 6.3.1), $x = A\sin 2t + B\cos 2t$. To find a particular solution we try $u^*(t) = Ct + D$, and find $C = 1$ and $D = 1/4$. It follows that the general solution is $x = A\sin 2t + B\cos 2t + t + 1/4$. To find the solution with the given initial conditions we must determine A and B. The initial condition $x(\pi/2) = 0$ gives $A\sin\pi + B\cos\pi + \pi/2 + 1/4 = 0$. Since $\sin\pi = 0$ and $\cos\pi = -1$, we find $B = \pi/2 + 1/4$. Since $\dot{x} = 2A\cos 2t - 2B\sin 2t + 1$, the equation $\dot{x}(\pi/2) = 0$ gives $-2A + 1 = 0$. Therefore $A = 1/2$, and so $x = \frac{1}{2}\sin 2t + (\frac{1}{4} + \frac{1}{2}\pi)\cos t + t + \frac{1}{4}$.

6.3.4 Since the right-hand side is a linear function of t, we try to find a particular solution of the form $u^* = Pt + Q$. We get $\dot{u}^* = P$ and $\ddot{u}^* = 0$, so we must have

$$\gamma[\beta + \alpha(1-\beta)]P - \gamma\delta^* Pt - \gamma\delta^* Q = -\gamma\delta^* kt - \gamma\delta^* L_0$$

It follows that $P = k$ and $Q = L_0 + [\beta + \alpha(1-\beta)]k/\delta^*$.

The characteristic equation is $r^2 + \gamma[\beta + \alpha(1-\beta)]r - \gamma\delta^* = 0$, with roots

$$r = -\tfrac{1}{2}\gamma[\beta + \alpha(1-\beta)] \pm \tfrac{1}{2}\sqrt{\gamma^2[\beta + \alpha(1-\beta)]^2 + 4\gamma\delta^*}$$

Oscillations occur if the characteristic roots are complex, i.e. if and only if $\frac{1}{4}\gamma^2[\beta+\alpha(1-\beta)]^2+\gamma\delta^* < 0$.

6.3.8 (a) This is an Euler differential equation. With $x = t^r$, we get $\dot{x} = rt^{r-1}$, and $\ddot{x} = r(r-1)t^{r-2}$. Inserting this into the given equation and cancelling t^r gives the equation $r^2 + 4r + 3 = 0$, with roots $r = -1$ and $r = -3$. The general solution is therefore $x = At^{-1} + Bt^{-3}$.

(b) Substituting $x = t^r$ into the homogeneous equation yields the equation $r^2 - 4r + 3 = (r-1)(r-3)$. The general solution of the homogeneous equation is therefore $x = At + Bt^3$. To find a particular solution of the nonhomogeneous equation, $t^2\ddot{x} - 3t\dot{x} + 3x = t^2$, we try $u^*(t) = Pt^2 + Qt + R$. Then $\dot{u}^* = 2Pt + Q$, and $\ddot{u}^* = 2P$. Inserted into the given equation this yields

$$2Pt^2 - 3t(2Pt + Q) + 3Pt^2 + 3Qt + 3R = t^2 \quad\iff\quad -Pt^2 + 3R = t^2$$

This holds for all t if and only if $P = -1$ and $R = 0$. (Note that Q did not appear in the last equation. That is because Qt is a solution of the homogeneous equation.) One particular solution of the nonhomogeneous equation is therefore $u^* = -t^2$, and so the general solution is $x = At + Bt^3 - t^2$.

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6.3.10 By Leibniz's rule, $\ddot{p} = a[D(p(t)) - S(p(t))] = a(d_1 - s_1)p + a(d_0 - s_0)$. In other words, p must satisfy $\ddot{p} + a(s_1 - d_1)p = a(d_0 - s_0)$. Note that $a(s_1 - d_1) > 0$, and see the answer in the book.

6.4

6.4.3 Write the equation as $\ddot{p}(t) - \lambda p(t) = k$, where $\lambda = \gamma(a - \alpha)$. If $\lambda > 0$, then the solution is $p(t) = Ae^{rt} + Be^{-rt} - k/r^2$, where $r = \sqrt{\lambda}$; if $\lambda = 0$, then the solution is $p(t) = At + B + \frac{1}{2}kt^2$; if $\lambda < 0$, then the solution is $p(t) = A\cos\sqrt{-\lambda}\,t + B\sin\sqrt{-\lambda}\,t - k/\lambda$.

The equation is not stable for any values of the constants. This is obvious from the form of the solutions—if $\lambda \geq 0$, the corresponding homogeneous equation has solutions that run off to infinity, and if $\lambda < 0$ the solutions oscillate with a fixed amplitude.

We can also see the instability from the criterion in (6.4.2): The characteristic equation is $r^2 - \lambda = 0$. If $\lambda > 0$, this equation has two real solutions, one positive and one negative. If $\lambda = 0$, the characteristic roots are $r_1 = r_2 = 0$. If $\lambda < 0$, the equation has complex roots with real part equal to 0.

6.5

6.5.2 (a) If we add the two equations we get $\dot{x} + \dot{y} = (a + b)(x + y)$. Hence $x + y = Ce^{(a+b)t}$. Because of the initial conditions $x(0) = \frac{1}{2}$, $y(0) = \frac{1}{2}$, we must have $C = 1$. This implies $\dot{x} = a(x + y) = ae^{(a+b)t}$ and $\dot{y} = be^{(a+b)t}$.

If $a + b \neq 0$, then

$$x = \frac{ae^{(a+b)t}}{a+b} + A, \qquad y = \frac{be^{(a+b)t}}{a+b} + B$$

for suitable constants A and B. The initial conditions yield

$$A = x(0) - \frac{a}{a+b} = \frac{1}{2} - \frac{a}{a+b} = \frac{b-a}{2(a+b)}, \qquad B = y(0) - \frac{b}{a+b} = \frac{1}{2} - \frac{b}{a+b} = \frac{a-b}{2(a+b)}$$

and so

$$x(t) = \frac{2ae^{(a+b)t} + b - a}{2(a+b)}, \qquad y(t) = \frac{2be^{(a+b)t} + a - b}{2(a+b)}$$

If $a + b = 0$, then $\dot{x} = a$ and $\dot{y} = b = -a$, and therefore

$$x = \tfrac{1}{2} + at, \qquad y = \tfrac{1}{2} - at$$

(b) The first equation gives $y = \dot{x} - 2tx$, and if we use this in the second equation we get

$$\ddot{x} - 2x - 2t\dot{x} = -2t - 2x \iff \ddot{x} - 2t\dot{x} = -2t$$

This is a first-order linear equation for $\dot{x}$ with general solution $\dot{x} = Ce^{-t^2} + 1$. From the first equation and the initial conditions we get $\dot{x}(0) = 0 + y(0) = 1$, so $C = 0$. Therefore $\dot{x} = 1$. Because $x(0) = 1$, we get $x = t + 1$, and $y = \dot{x} - 2tx = 1 - 2t(t + 1) = -2t^2 - 2t + 1$.

(c) The first equation yields $y = -\frac{1}{2}\dot{x} + \frac{1}{2}\sin t$. From the second equation we then get

$$-\tfrac{1}{2}\ddot{x} + \tfrac{1}{2}\cos t = 2x + 1 - \cos t \iff \ddot{x} + 4x = -2 + 3\cos t \qquad (*)$$

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The corresponding homogeneous equation, $\ddot{x}+4x = 0$, has the general solution $x = A\cos 2t + B\sin 2t$. To find a particular solution of $(*)$ we try $u^* = C + D\cos t + E\sin t$. Then $\ddot{u}^* = -D\cos t - E\sin t$ and $\ddot{u}^* + 4u^* = 4C + 3D\cos t + 3E\sin t$. It follows that $C = -1/2$, $D = 1$, and $E = 0$. Thus the general solution of $(*)$ is $x = A\cos 2t + B\sin 2t - \frac{1}{2} + \cos t$, and we get $y = -\frac{1}{2}\dot{x} + \frac{1}{2}\sin t = -A\cos 2t + B\sin 2t + \sin t$. The initial conditions $x(0) = y(0) = 0$ yield $B - \frac{1}{2} + 1 = 0$ and $-A = 0$, so the solutions we are looking for are

$$x = -\tfrac{1}{2}\cos 2t + \cos t - \tfrac{1}{2}, \qquad y = -\tfrac{1}{2}\sin 2t + \sin t$$

6.5.3 The first equation gives $p = e^{-2t}(\dot{x}-x)$. Then $\dot{p} = -2e^{-2t}(\dot{x}-x)+e^{-2t}(\ddot{x}-\dot{x}) = e^{-2t}(\ddot{x}-3\dot{x}+2x)$. If we insert these expressions for p and $\dot{p}$ into the second equation, we get

$$e^{-2t}(\ddot{x} - 3\dot{x} + 2x) = 2e^{-2t}x - e^{-2t}(\dot{x} - x) = e^{-2t}(3x - \dot{x}) \quad\Longrightarrow\quad \ddot{x} - 2\dot{x} - x = 0$$

The general solution of the last equation is $\underline{x = Ae^{(1+\sqrt{2})t} + Be^{(1-\sqrt{2})t}}$, where $1\pm\sqrt{2}$ are the roots of the characteristic equation, $r^2 - 2r - 1 = 0$. A straightforward calculation gives $\dot{x} - x = A\sqrt{2}e^{(1+\sqrt{2})t} - B\sqrt{2}e^{(1-\sqrt{2})t}$ and then $\underline{p} = e^{-2t}(\dot{x} - x) = \underline{A\sqrt{2}e^{(\sqrt{2}-1)t} - B\sqrt{2}e^{(-\sqrt{2}-1)t}}$.

6.5.4 From the first equation, $\sigma = \alpha\pi - \dot{\pi}$. Inserting this into the second equation, we get

$$\alpha\dot{\pi} - \ddot{\pi} = \pi - \frac{\alpha}{\beta}\pi + \frac{1}{\beta}\dot{\pi} \quad\Longleftrightarrow\quad \ddot{\pi} + \Big(\frac{1}{\beta} - \alpha\Big)\dot{\pi} + \Big(1 - \frac{\alpha}{\beta}\Big)\pi = 0 \qquad (*)$$

which is a second-order differential equation for π with constant coefficients. The characteristic equation is $r^2+(1/\beta-\alpha)r+(1-\alpha/\beta) = 0$, with roots $r_{1,2} = \frac{1}{2}(\alpha-1/\beta)\pm\frac{1}{2}\sqrt{(\alpha+1/\beta)^2-4}$. If $\alpha+1/\beta > 2$, then r_1 and r_2 are real and different, and the general solution of $(*)$ is $\underline{\pi = Ae^{r_1t} + Be^{r_2t}}$. It follows that $\underline{\sigma = \alpha\pi - \dot{\pi} = (\alpha - r_1)Ae^{r_1t} + (\alpha - r_2)Be^{r_2t}}$.

6.6

6.6.3 (ii) The equilibrium point of the linear system $\dot{x} = x + 2y$, $\dot{y} = -y$, is $(x^*, y^*) = (-5, 2)$. Let $z = x + 5$ and $w = y - 2$ (cf. example 6.5.4). The given equation system is equivalent to the system

$$\begin{aligned}\dot{w} &= w + 2z\\ \dot{z} &= -z\end{aligned} \quad\Longleftrightarrow\quad \begin{pmatrix}\dot{w}\\ \dot{z}\end{pmatrix} = \mathbf{A}\begin{pmatrix}w\\ z\end{pmatrix} \qquad (*)$$

with coefficient matrix $\mathbf{A} = \begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$. Since the trace of $\mathbf{A}$ is 0, the system is not globally asymptotically stable. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = -1$, with corresponding eigenvectors $\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\begin{pmatrix}1\\ -1\end{pmatrix}$. According to (6.5.9), the solution of $(*)$ is $\begin{pmatrix}w\\ z\end{pmatrix} = Ae^t\begin{pmatrix}1\\ 0\end{pmatrix} + Be^{-t}\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}Ae^t + Be^{-t}\\ -Be^{-t}\end{pmatrix}$. The solution of the given system is therefore $x = z - 5 = Ae^t + Be^{-t} - 5$ and $y = w + 2 = -Be^{-t} + 2$, the same solution as in (i).

It is clear that if $A \neq 0$, then x does not converge to the equilibrium value as $t \to \infty$, which confirms that the system is not asymptotically stable.

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6.6.4 (a) From the first equation, $y = \frac{1}{2}(\dot{x} - ax - \alpha)$. The second equation then yields

$$\tfrac{1}{2}(\ddot{x} - a\dot{x}) = 2x + \tfrac{1}{2}a(\dot{x} - ax - \alpha) + \beta \quad \Longleftrightarrow \quad \ddot{x} - 2a\dot{x} + (a^2 - 4)x = 2\beta - \alpha a \qquad (\diamond)$$

The characteristic equation is $r^2 - 2ar + a^2 - 4 = 0$, which has the roots $r_{1,2} = a \pm 2$. It follows that the homogeneous equation associated with ($\diamond$) has the general solution $x_{\mathrm{H}} = Ae^{(a-2)t} + Be^{(a+2)t}$. Hence, the general solution of equation ($\diamond$) itself is $x_{\mathrm{H}} + u^*$, where u^* is any particular solution of ($\diamond$). Since the right-hand side of the equation is a constant, we look for a suitable constant u^*, which turns out to be $u^* = (2\beta - \alpha a)/(a^2 - 4)$. Thus, the general solution of ($\diamond$) is $\underline{x = Ae^{(a-2)t} + Be^{(a+2)t} + (2\beta - \alpha a)/(a^2 - 4)}$. Then the equation $y = \frac{1}{2}(\dot{x} - ax - \alpha)$ yields $\underline{y = -Ae^{(a-2)t} + Be^{(a+2)t} + (2\alpha - a\beta)/(a^2 - 4)}$.

(b) The equilibrium point (x^*, y^*) is given by the equation system

$$ax^* + 2y^* + \alpha = 0, \qquad 2x^* + ay^* + \beta = 0$$

Easy calculations show that $x^* = (2\beta - \alpha a)/(a^2 - 4)$ and $y^* = (2\alpha - a\beta)/(a^2 - 4)$. Of course, this is just the stationary solution of the system in part (a), given by $A = B = 0$. The equilibrium point is globally asymptotically stable if and only if $e^{(a-2)t}$ and $e^{(a+2)t}$ both tend to 0 as $t \to \infty$, and this happens if and only if both $a - 2$ and $a + 2$ are negative, i.e. if and only if $a < -2$.

An alternative way to check stability is to consider the trace and determinant of the coefficient matrix $\mathbf{A} = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$. We get $\operatorname{tr}(\mathbf{A}) = 2a$ and $|\mathbf{A}| = a^2 - 4$, and Theorem 6.6.1 says that the equilibrium point is globally asymptotically stable if and only if $2a < 0$ and $a^2 - 4 > 0$, which is equivalent to $a < -2$.

(c) With $a = -1$, $\alpha = -4$, and $\beta = -1$, the general solution of the system is $x = Ae^{-3t} + Be^t + 2$, $y = -Ae^{-3t} + Be^t + 3$. It is clear that this will converge to the equilibrium point $(2, 3)$ if and only if $B = 0$. One such solution is then $x = e^{-3t} + 2$, $y = -e^{-3t} + 3$. This solution moves towards $(2, 3)$ along the line $x + y = 5$, and so do all the convergent solutions, i.e. the solutions with $B = 0$.

6.7

6.7.4 (a) See Figs. A6.7.4(a) and (b) in the answer section of the book. The system has a single equilibrium point, namely $(0, 0)$. It seems clear from the phase diagram that this is not a stable equilibrium. Indeed, the equations show that any solution through a point on the y-axis below the origin will move straight downwards, away from $(0, 0)$.

(b) The first equation has the general solution $x = Ae^{-t}$, and the initial condition $x(0) = -1$ implies $A = -1$, so $x = -e^{-t}$. The second equation is the system the becomes $\dot{y} = e^{-t}y - y^2$, which is a Bernoulli equation.

With $z = y^{1-2} = y^{-1} = 1/y$ we get $y = 1/z$, $\dot{y} = -\dot{z}/z^2$, and

$$-\dot{z}/z^2 = e^{-t}(1/z) - 1/z^2 \quad \Longrightarrow \quad \dot{z} + e^{-t}z = 1$$

If we use formula (5.4.7) with $a(t) = e^{-t}$, $b(t) = 1$, $t_0 = 1$, and $z_0 = 1$, we get $-\int_s^t a(\xi)\,d\xi = e^{-t} - e^{-s}$ and

$$z = e^{e^{-t}-1} + \int_0^t e^{e^{-t}-e^{-s}}\,ds = e^{e^{-t}}\Big(e^{-1} + \int_0^t e^{-e^{-s}}\,ds\Big)$$

$$y = \frac{e^{-e^{-t}}}{e^{-1} + \int_0^t e^{-e^{-s}}\,ds}$$

For all $s \geq 0$ we have $e^{-s} \leq 1$, so $-e^{-s} \geq -1$ and $e^{-e^{-s}} \geq e^{-1}$. It follows that for $t > 0$ we have $\int_0^t e^{-e^{-s}}\,ds \geq e^{-1}t$, $e^{-e^{-t}} \leq e^0 = 1$ and $y(t) \leq 1/(e^{-1} + te^{-1}) \to 0$ as $t\infty$.

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6.7.6 The equation $\dot{x} = -x$ has the general solution $x = Ae^{-t}$. The initial condition $x(0) = 1$ then implies $x(t) = e^{-t}$. The second equation becomes $\dot{y} = -e^{-2t}y$. This is a separable equation and we get

$$\int \frac{dy}{y} = -\int e^{-2t}\,dt \quad\Longrightarrow\quad \ln|y| = \frac{1}{2}e^{-2t} + C$$

The initial condition $y(0) = 1$ yields $C = -\frac{1}{2}$, and we get $y(t) = e^{(e^{-2t}-1)/2}$. As $t \to \infty$, the point $(x(t), y(t))$ tends to $(0, e^{-1/2})$.

6.8

6.8.5 Let $f(Y, K) = \alpha\big(I(Y, K) - S(Y, K)\big)$ and $g(Y, K) = I(Y, K)$. The matrix $\mathbf{A}$ in Theorem 6.8.2 is $\mathbf{A}(Y, K) = \begin{pmatrix} f'_Y & f'_K \\ g'_Y & g'_K \end{pmatrix} = \begin{pmatrix} \alpha(I'_Y - S'_Y) & \alpha(I'_K - S'_K) \\ I'_Y & I'_K \end{pmatrix}$. Now, $\mathrm{tr}\big(\mathbf{A}(Y, K)\big) = \alpha(I'_Y - S'_Y) + I'_K < 0$ and $|\mathbf{A}(Y, K)| = \alpha(I'_Y - S'_Y)I'_K - \alpha(I'_K - S'_K)I'_Y = \alpha(I'_Y S'_K - I'_K S'_Y) > 0$ by the assumptions in the problem. Finally, $f'_Y g'_K = \alpha(I'_K - S'_K)I'_Y < 0$ everywhere, so an equilibrium point for the system must be globally asymptotically stable according to Olech's theorem.

6.8.6 See the answer in the book for the stability question.

K must satisfy the Bernoulli equation $\dot{K} = sK^{\alpha} - \delta K$. This corresponds to the equation in Problem 5.6.4, with $\gamma_1 b = s$ and $\gamma_2 = -\delta$. The general solution is therefore $K(t) = \big[Ce^{-\delta(1-\alpha)t} + s/\delta\big]^{1/(1-\alpha)}$. The initial condition $K(0) = K_0$ yields $C = K_0^{1-\alpha} - s/\delta$. Since $\delta(1-\alpha) > 0$, the solution $K(t) = \big[(K_0^{1-\alpha} - s/\delta)e^{-\delta(1-\alpha)t} + s/\delta\big]^{1/(1-\alpha)}$ tends to $(s/\delta)^{1/(1-\alpha)} = K^*$ as $t \to \infty$.

6.9

6.9.2 Write the system as

$$\dot{k} = F(k, c) = f(k) - \delta k - c, \qquad \dot{c} = G(k, c) = -c(r + \delta - f'(k))$$

The Jacobian matrix at (k, c) is

$$\mathbf{A}(k, c) = \begin{pmatrix} F'_k & F'_c \\ G'_k & G'_c \end{pmatrix} = \begin{pmatrix} f'(k) - \delta & -1 \\ cf''(k) & f'(k) - r - \delta \end{pmatrix}$$

The origin $(k, c) = (0, 0)$ is an equilibrium point, but not the one the authors had in mind. At an equilibrium point (k^*, c^*) with $c^* > 0$ we have $c^* = f(k^*) - \delta k^*$ and $f'(k^*) = r + \delta$. Then

$$\mathbf{A}(k^*, c^*) = \begin{pmatrix} f'(k^*) - \delta & -1 \\ cf''(k^*) & 0 \end{pmatrix} = \begin{pmatrix} r & -1 \\ cf''(k^*) & 0 \end{pmatrix}$$

has determinant $|\mathbf{A}(k^*, c^*)| = cf''(k^*) < 0$, so (k^*, c^*) is a saddle point.

6.9.3 (a) $(x_0, y_0) = (4/3,\ 8/3)$. It is a saddle point because the Jacobian at $(4/3,\ 8/3)$ is $\mathbf{A} = \begin{pmatrix} y - x - 2 & x \\ y^2/2x^2 & 1 - y/x \end{pmatrix} = \begin{pmatrix} -2/3 & 4/3 \\ 2 & -1 \end{pmatrix}$, with determinant $|\mathbf{A}| = -2$.

(b) See Fig. A6.9.3 in the answer section of the book.

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6.9.4 (a) The equilibrium points are the solutions of the equation system

$$\text{(i)}\quad y^2 - x = 0, \qquad \text{(ii)}\quad 25/4 - y^2 - (x - 1/4)^2 = 0$$

Substituting x for y^2 in (ii) leads to the quadratic equation $x^2 + x/2 - 99/16 = 0$, which has the solutions $x = (-1 \pm 10)/4$. Since $x = y^2$ must be nonnegative, we get $x = 9/4$ and $y = \pm 3/2$. The Jacobian at (x, y) is $\mathbf{A}(x, y) = \begin{pmatrix} -1 & 2y \\ -2x + 1/2, & -2y \end{pmatrix}$, so $\mathbf{A}(9/4, 3/2) = \mathbf{A}_1 = \begin{pmatrix} -1 & 3 \\ -4 & -3 \end{pmatrix}$ and $\mathbf{A}(9/4, -3/2) = \mathbf{A}_2 = \begin{pmatrix} -1 & -3 \\ -4 & 3 \end{pmatrix}$. The determinants and traces of these matrices are $|\mathbf{A}_1| = 15$, $\operatorname{tr}(\mathbf{A}_1) = -4$, $|\mathbf{A}_2| = -15$, and $\operatorname{tr}(\mathbf{A}_2) = 2$. It follows that $(9/4, 3/2)$ is locally asymptotically stable, whereas $(9/4, -3/2)$ is a saddle point. These conclusions are confirmed by the solution to part (b). See Fig. A6.9.4 in the answer section of the book.

7 Differential Equations III: Higher-Order Equations

7.1

7.1.3 The homogeneous equation $\ddot{x} + x = 0$ has the two linearly independent solutions $u_1 = \sin t$ and $u_2 = \cos t$. To find the solution of the equation $\ddot{x} + x = 1/t$ we use the method of variation of parameters. Let $x = C_1(t)\sin t + C_2(t)\cos t$. The two equations to determine $\dot{C}_1(t)$ and $\dot{C}_2(t)$ are

$$\dot{C}_1(t)\sin t + \dot{C}_2(t)\cos t = 0$$
$$\dot{C}_1(t)\cos t - \dot{C}_2(t)\sin t = 1/t$$

This is a linear equation system in the unknowns $\dot{C}_1(t)$ and $\dot{C}_2(t)$, and the determinant of the system is $\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1$. (See Section B.1.) Cramer's rule gives

$$\dot{C}_1(t) = \frac{\cos t}{t}, \qquad \dot{C}_2(t) = -\frac{\sin t}{t}$$

It follows that $u^*(t) = \sin t \int \frac{\cos t}{t}\,dt - \cos t \int \frac{\cos t}{t}\,dt$ is the general solution of the given equation (provided we include an arbitrary constant of integration in each of the two integrals).

7.2

7.2.2 Integer roots of the characteristic equation $r^3 - r^2 - r + 1 = 0$ must divide the constant term 1 (EMEA, Note 4.7.2), so the only possibilities are ± 1, and we see that both $r = -1$ and $r = 1$ are roots. Moreover, $(r^3 - r^2 - r + 1) \div (r - 1) = (r^2 - 1)$, so $r^3 - r^2 - r + 1 = (r - 1)(r^2 - 1) = (r - 1)^2(r + 1)$. According to the general method for finding linearly independent solutions to linear homogeneous equations with constant coefficients, the general solution of the associated homogeneous equation is $x_{\mathrm{H}} = (A + Bt)e^t + Ce^{-t}$. Looking at the right-hand side of the given nonhomogeneous equation, it might seem natural to try $u^* = (D + Et)e^{-t}$ as a particular solution. But that does not work because $r = -1$ is a root in the characteristic equation. We must therefore increase the degree of the polynomial factor and try with a quadratic polynomial instead. Let $u^* = (Et + Ft^2)e^{-t}$. A bit of tedious algebra gives $\dot{u}^* = (E + (2F - E)t - Ft^2)e^{-t}$, $\ddot{u}^* = (2F - 2E + (E - 4F)t + Ft^2)e^{-t}$, $\dddot{u}^* = (3E - 6F + (6F - E)t - Ft^2)e^{-t}$,

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and finally $\dddot{u}^* - \ddot{u}^* - \dot{u}^* + u^* = (4E - 8F + 8Ft)e^{-t}$. This equals $8te^{-t}$ if $F = 1$ and $E = 2$, so the general solution of the given equation is $x = x_{\mathrm{H}} + u^* = (A + Bt)e^t + (C + 2t + t^2)e^{-t}$. Requiring this solution to satisfy the initial conditions gives $A = -1$, $B = 1$, $C = 1$, and so $x = (t-1)e^t + (t+1)^2e^{-t}$.

Why did we not include a term of the form De^{-t} in the tentative solution u^*? The answer is that De^{-t} is a solution of the homogeneous equation for every value of D, so D would not appear in $\dddot{u}^* - \ddot{u}^* - \dot{u}^* + u^*$.

7.2.3 Differentiating the equation w.r.t. t gives (using the product rule and (4.1.5))

$$\dddot{K} = (\gamma_1\kappa + \gamma_2)\ddot{K} + (\gamma_1\sigma + \gamma_3)\mu_0\mu e^{\mu t}\int_0^t e^{-\mu\tau}\dot{K}(\tau)d\tau + (\gamma_1\sigma + \gamma_3)\mu_0 e^{\mu t}e^{-\mu t}\dot{K}(t) \qquad (*)$$

From the given equation, $(\gamma_1\sigma + \gamma_3)\mu_0 e^{\mu t}\int_0^t e^{-\mu\tau}\dot{K}(\tau)d\tau = \ddot{K} - (\gamma_1\kappa + \gamma_2)\dot{K}$. Inserting this into $(*)$ yields the equation $\dddot{K} - p\ddot{K} + q\dot{K} = 0$ given in the answer in the book. One root of the characteristic equation $r^3 - pr^2 + qr = 0$ is $r_3 = 0$. The other two are the roots r_1 and r_2 of $r^2 - pr + q = 0$, and they are real, nonzero, and different if $p^2 - 4q > 0$ and $q \neq 0$. If these conditions are satisfied, it follows from the theory in this section that the general solution of the differential equation is of the form $K(t) = C_1e^{r_1t} + C_2e^{r_2t} + C_3e^{r_3t}$.

7.3

7.3.2 The roots of the characteristic equation $r^3 + 4r^2 + 5r + 2 = 0$ are $r_1 = r_2 = -1$ and $r_3 = -2$, which are all negative, so global asymptotic stability also follows from Theorem 7.3.1.

7.4

7.4.1 (i) The system is: (a) $\dot{x}_1 = -x_1 + x_2 + x_3$, (b) $\dot{x}_2 = x_1 - x_2 + x_3$, (c) $\dot{x}_3 = x_1 + x_2 + x_3$. Differentiating (a) w.r.t. t and inserting from (b) and (c) gives (d) $\ddot{x}_1 + \dot{x}_1 - 2x_1 = 2x_3$. Differentiating once more w.r.t. t and inserting from (c) gives $\dddot{x}_1 + \ddot{x}_1 - 2\dot{x}_1 = 2\dot{x}_3 = 2x_1 + 2(x_2 + x_3) = 2x_1 + 2(\dot{x}_1 + x_1)$, using (a) again. Thus the differential equation for x_1 is (e) $\dddot{x}_1 + \ddot{x}_1 - 4\dot{x}_1 - 4x_1 = 0$. Since the characteristic polynomial is $(r+1)(r+2)(r-2)$, the general solution is $x_1 = C_1e^{-t} + C_2e^{-2t} + C_3e^{2t}$. From (d) we find $x_3 = \frac{1}{2}(\ddot{x}_1 + \dot{x}_1 - 2x_1) = -C_1e^{-t} + 2C_3e^{2t}$. We then find x_2 from (a): $x_2 = \dot{x}_1 + x_1 - x_3 = C_1e^{-t} - C_2e^{-2t} + C_3e^{2t}$.

(ii) We write the system as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. The eigenvalues of $\mathbf{A}$ are the solutions of the equation $\begin{vmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$, and we find them to be $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = 2$. (These are the same as the solutions of the characteristic equation of the differential equation (e). This is no coincidence. See the remark above Theorem 6.6.1 concerning second-order systems.) The eigenvectors associated with the eigenvalues are determined by the three systems

$$\begin{aligned} x_2 + x_3 &= 0 \\ x_1 \quad\quad + x_3 &= 0\,, \\ x_1 + x_2 + 2x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0\,, \\ x_1 + x_2 + 3x_3 &= 0 \end{aligned} \qquad \begin{aligned} -3x_1 + x_2 + x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

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The following vectors are solutions of these systems:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

The solution of the given system of differential equations is then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

We see that we have arrived at the same solutions as above.

(iii) The resolvent, with $t_0 = 0$, is

$$\mathbf{P}(t, 0) = \begin{pmatrix} \frac{1}{3}e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{6}e^{2t} & \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{6}e^{2t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t} + \frac{1}{6}e^{2t} & \frac{1}{3}e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{6}e^{2t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} \\ -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} & \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \end{pmatrix}$$

The ith column of $\mathbf{P}(t, 0)$ is a vector $(x_1(t), x_2(t), x_3(t))'$ of particular solutions of the system such that $(x_1(0), x_2(0), x_3(0))'$ is the ith standard unit vector $\mathbf{e}_i$. In particular, $\mathbf{P}(0, 0) = \mathbf{I}_3$. It is not hard to verify that $\mathbf{AP}(t, 0) = (d/dt)\mathbf{P}(t, 0)$.

7.5

7.5.4 (a) Equation $(*)$ is separable, so $\displaystyle\int \frac{g_1(x)}{f_1(x)}\,dx = \int \frac{f_2(y)}{g_2(y)}\,dy + C$ for some constant C. Therefore, the function $H(x, y)$ is constant along each solution curve for the system.

(b) $\displaystyle H(x, y) = \int \frac{bx - h}{x}\,dx - \int \frac{k - ay}{y}\,dy = \int \Big(b - \frac{h}{x}\Big)\,dx - \int \Big(\frac{k}{y} - a\Big)\,dy =$

$$bx - h\ln x - (k\ln y - ay) + C = b(x - x_0\ln x) + a(y - y_0\ln y) + C,$$

where $x_0 = h/b$ and $y_0 = k/a$.

7.5.5 We can write the system as

$$\dot{x} = x(k - ay - \varepsilon x), \qquad \dot{y} = y(-h + bx - \delta y)$$

It is clear that a point (x_0, y_0) with $x_0 \neq 0$ and $y_0 \neq 0$ is an equilibrium point if and only if

$$\begin{aligned} bx_0 - \delta y_0 &= h \\ \varepsilon x_0 + ay_0 &= k \end{aligned} \qquad (*)$$

The determinant of this system is $ab + \delta\varepsilon \neq 0$, so it has a unique solution, and Cramer's rule gives the solution as $x_0 = (ah + k\delta)/(ab + \delta\varepsilon)$, $y_0 = (bk - h\varepsilon)/(ab + \delta\varepsilon)$, as given in the problem.

We claim that the function

$$L(x, y) = b(x - x_0\ln x) + a(y - y_0\ln y) - b(x_0 - x_0\ln x_0) + a(y_0 - y_0\ln y_0)$$

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is a strong Liapunov function for the system, with (x_0, y_0) as its minimum point. First note that $L(x_0, y_0) = 0$. Moreover, $L'_x(x, y) = b(1 - x_0/x)$ and $L'_y(x, y) = a(1 - y_0/y)$ are both 0 at (x_0, y_0) and $L''_{xx} = bx_0/x^2 > 0$, $L''_{yy} = ay_0/y^2 > 0$, and $L''_{xy} = 0$, so $L(x, y)$ is strictly convex and has a unique minimum at (x_0, y_0). Finally,

$$\begin{aligned}\dot{L} &= b\big(1 - x_0/x\big)\dot{x} + a\big(1 - y_0/y\big)\dot{y} = b(k - ay - \varepsilon x)(x - x_0) + a(-h + bx - \delta y)(y - y_0)\\ &= b(\varepsilon x_0 + ay_0 - ay - \varepsilon x)(x - x_0) + a(\delta y_0 - bx_0 + bx - \delta y)(y - y_0)\\ &= -\varepsilon b(x - x_0)^2 - a\delta(y - y_0)^2\end{aligned}$$

which is negative for $(x, y) \neq (x_0, y_0)$. We used the equations $k = \varepsilon x_0 + ay_0$ and $h = -\delta y_0 + bx_0$ from $(*)$. We have proved that L is a strong Liapunov function for the system, so (x_0, y_0) is locally asymptotically stable.

7.7

7.7.1 (a) By integration, $z = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 - e^x y + \varphi(y)$, where $\varphi(y)$ is an arbitrary function and plays the role of a constant of integration when we integrate with respect to x.

(b) The recipe for solving equations of the form (7.7.2) leads to the solution $z = 3x + \varphi(y - 2x)$, where φ is an arbitrary differentiable function. It looks as if x has a special role here, but that is an illusion. If we use the recipe with y instead of x as the independent variable in equations (7.7.3), we are led to $z = 3y/2 + \psi(x - y/2)$, where y seems to be singled out. Actually, these solutions are just two ways of writing the same thing. The functions φ and ψ are related by the equation $\psi(u) = \varphi(-2u) + 3u$.

(c) The equations in (7.7.3) are both separable, $dy/dx = y^2/x^2$ and $dz/dx = z^2/x^2$. The solutions are $-1/y = -1/x + C_1$, $-1/z = -1/x + C_2$. The general solution is therefore $\Phi(1/x - 1/y, 1/x - 1/z) = 0$, or $1/z = 1/x - \varphi(1/x - 1/y)$, and hence $z = \dfrac{x}{1 - x\varphi(1/x - 1/y)}$, where φ is an arbitrary differentiable function.

7.7.3 (a) The equations in (7.7.3) are $dy/dx = -y/x$ and $dz/dx = 1$. The latter equation gives $z = x + C_1$, and so $z - x = C_1$. The first equation is separable with solution $xy = C_2$. The general solution of $(*)$ is given by $\Phi(z - x, xy) = 0$. Solving this equation for the first variable yields $z = x + \varphi(xy)$, where φ is an arbitrary differentiable function.

(b) The condition $f(x, 1) = x^2$ implies that $x + \varphi(x) = x^2$. Thus $\varphi(x) = -x + x^2$ for all x, and hence $f(x, y) = x + \varphi(xy) = x - xy + x^2y^2$.

7.7.7 The equations in (7.7.3) are $dv_2/dv_1 = v_2/v_1$ and $dx/dv_1 = x\varepsilon(x)/v_1$, with the solutions $v_2/v_1 = C_1$, $f(x) - \ln v_1 = C_2$, where $f(x) = \int(1/x\varepsilon(x))\,dx$. Since $f'(x) = 1/x\varepsilon(x) > 0$, f is strictly increasing, and has an inverse f^{-1} that is also strictly increasing. The general solution is $\Phi(v_2/v_1, f(x) - \ln v_1) = 0$, or $f(x) = \ln v_1 + \varphi(v_2/v_1)$. Hence, $x = f^{-1}(\ln v_1 + \varphi(v_2/v_1))$. Define $g(v_1, v_2) = e^{\ln v_1 + \varphi(v_2/v_1)} = v_1 e^{\varphi(v_2/v_1)}$. Then g is homogeneous of degree 1, and we see that $x = f^{-1}(\ln(g(v_1, v_2))$. The composition F of the two increasing functions f^{-1} and ln is increasing. It follows that $x = F(g(v_1, v_2))$ is homothetic.

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8 Calculus of Variations

8.2

8.2.3 (a) With $F(t,x,\dot{x}) = x^2 + \dot{x}^2 + 2xe^t$ we get $F'_x = 2x + 2e^t$ and $F'_{\dot{x}} = 2\dot{x}$, and the Euler equation becomes

$$2x + 2e^t - \frac{d}{dt}(2\dot{x}) = 0 \iff 2x + 2e^t - 2\ddot{x} = 0 \iff \ddot{x} - x = e^t$$

(b) With $F(t,x,\dot{x}) = -e^{\dot{x}-ax}$ we get $F'_x = ae^{\dot{x}-ax}$ and $F'_{\dot{x}} = -e^{\dot{x}-ax}$. The Euler equation is

$$ae^{\dot{x}-ax} + \frac{d}{dt}e^{\dot{x}-ax} = 0 \iff ae^{\dot{x}-ax} + e^{\dot{x}-ax}(\ddot{x} - a\dot{x}) = 0 \iff \ddot{x} - a\dot{x} + a = 0$$

(c) Here $F(t,x,\dot{x}) = [(x-\dot{x})^2 + x^2]e^{-at}$, so $F'_x = [2(x-\dot{x}) + 2x]e^{-at}$ and $F'_{\dot{x}} = -2(x-\dot{x})e^{-at}$. The Euler equation becomes

$$\begin{aligned}
&[2(x-\dot{x}) + 2x]e^{-at} + \frac{d}{dt}[2(x-\dot{x})e^{-at}] = 0 \\
&\qquad\iff [2(x-\dot{x}) + 2x]e^{-at} + 2(\dot{x}-\ddot{x})e^{-at} - 2a(x-\dot{x})e^{-at} = 0 \\
&\qquad\iff 2(x-\dot{x}) + 2x + 2(\dot{x}-\ddot{x}) - 2a(x-\dot{x}) = 0 \\
&\qquad\iff \ddot{x} - a\dot{x} + (a-2)x = 0
\end{aligned}$$

(d) With $F(t,x,\dot{x}) = 2tx + 3x\dot{x} + t\dot{x}^2$ we get $F'_x = 2t + 3\dot{x}$ and $F'_{\dot{x}} = 3x + 2t\dot{x}$, and the Euler equation becomes

$$2t + 3\dot{x} - \frac{d}{dt}(3x + 2t\dot{x}) = 0 \iff 2t + 3\dot{x} - 3\dot{x} - 2\dot{x} - 2t\ddot{x} = 0 \iff t\ddot{x} + \dot{x} = t$$

It is worth noting that this is an exact equation for $\dot{x}$ because it says that $(d/dt)(t\dot{x}) = t$. Hence, $t\dot{x} = \frac{1}{2}t^2 + C$, which implies $\dot{x} = \frac{1}{2}t + C/t$, and so $x = \frac{1}{4}t^2 + C\ln|t| + A$, where C and A are constants.

8.2.4 With $F(t,x,\dot{x}) = x^2 + 2tx\dot{x} + \dot{x}^2$ we get $F'_x = 2x + 2t\dot{x}$ and $F'_{\dot{x}} = 2tx + 2\dot{x}$, so the Euler equation (8.2.2) is

$$2x + 2t\dot{x} - \frac{d}{dt}(2tx + 2\dot{x}) = 0 \iff 2x + 2t\dot{x} - (2x + 2t\dot{x} + 2\ddot{x}) = 0 \iff \ddot{x} = 0$$

The general solution of $\ddot{x} = 0$ is $x = At + B$, and the boundary conditions $x(0) = 1$ and $x(1) = 2$ yield $A = B = 1$. Thus the only admissible function that satisfies the Euler equation is $x = t + 1$.

We have $F''_{xx} = 2 > 0$, $F''_{x\dot{x}} = 2t$, and $F''_{\dot{x}\dot{x}} = 2 > 0$, and since $F''_{xx}F''_{\dot{x}\dot{x}} - (F''_{x\dot{x}})^2 = 4 - 4t^2 \geq 0$ for all t in $[0,1]$, it follows that $F(t,x,\dot{x})$ is convex with respect to $(x,\dot{x})$ as long as $t \in [0,1]$. Hence, by Theorem 8.3.1, $x = t + 1$ is the optimal solution of the problem.

8.2.6 Let $F(t,x,\dot{x}) = \sqrt{1+\dot{x}^2}$. Then $F'_x = 0$ and $F'_{\dot{x}} = \dfrac{\dot{x}}{\sqrt{1+\dot{x}^2}}$. The Euler equation, $F'_x - \dfrac{d}{dt}F'_{\dot{x}} = 0$, therefore reduces to

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = 0, \qquad \text{i.e.} \qquad \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = C \quad \text{(constant)}$$

This implies

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$$\frac{\dot{x}^2}{1+\dot{x}^2} = C^2, \qquad \text{so} \qquad \dot{x} = C_1 \quad (= \frac{C}{\sqrt{1-C^2}})$$

Since $\dot{x}$ is a constant, x is a linear function of t:

$$x = C_1 t + C_2.$$

Thus the graph of x is a straight line, namely the straight line through the points (t_0, x_0) and (t_1, x_1). Of course, this is precisely what we would expect: the shortest curve between two points is the straight line segment between them. The function x is given by the equation

$$x(t) = x_0 + \frac{x_1 - x_0}{t_1 - t_0}(t - t_0)$$

Note also that $F''_{\dot{x}\dot{x}} = 1/(1+\dot{x}^2)^{3/2} > 0$, so F is convex with respect to $(x, \dot{x})$, and Theorem 8.3.1 in Section 8.3 shows that the function we have found really does give the minimum, at least among the admissible C^2 functions.

8.3

8.3.2 (a) The objective function is $F(t, x, \dot{x}) = U(\bar{c} - \dot{x}e^{rt})$, and

$$\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial F}{\partial \dot{x}} = -U'(\bar{c} - \dot{x}e^{rt})e^{rt}.$$

As $\partial F/\partial x = 0$, the Euler equation reduces to

$$\frac{d}{dt}\big(-U'(\bar{c} - \dot{x}e^{rt})e^{rt}\big) = 0, \quad \text{so} \quad U'(\bar{c} - \dot{x}e^{rt})e^{rt} = K \quad \text{(a constant)}$$

(Evaluating the derivative $\frac{d}{dt}\big(-U'(\bar{c} - \dot{x}e^{rt})e^{rt}\big)$ above leads to the equation

$$-U''(\bar{c} - \dot{x}e^{rt})(-\ddot{x}e^{rt} - r\dot{x}e^{rt})e^{rt} - rU'(\bar{c} - \dot{x}e^{rt})e^{rt} = 0$$

This can be simplified to

$$\ddot{x} + r\dot{x} = \frac{rU'(\bar{c} - \dot{x}e^{rt})}{U''(\bar{c} - \dot{x}e^{rt})}e^{-rt}$$

which will most likely be harder to solve.)

(b) It follows from part (a) that

$$U'(\bar{c} - \dot{x}e^{rt}) = Ke^{-rt}$$

If $U(c) = -e^{-vc}/v$, then $U'(c) = e^{-vc}$, and we get

$$\begin{aligned}
\exp(-v\bar{c} + v\dot{x}e^{rt}) &= Ke^{-rt} \\
-v\bar{c} + v\dot{x}e^{rt} &= \ln K - rt \\
\dot{x}e^{rt} &= C - rt/v \qquad (C = \bar{c} + (\ln K)/v) \\
\dot{x} &= (C - rt/v)e^{-rt}
\end{aligned}$$

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Integration by parts yields

$$x = x(t) = \int (C - rt/v)e^{-rt}\,dt = A + (B + t/v)e^{-rt}$$

where $B = (1 - vC)/rv$. Finally, the constants A and B are determined by the equations

$$A + B = x(0) = x_0, \qquad A + (B + T/v)e^{-rT} = x(T) = 0$$

U is concave because $U''(c) = -ve^{-vc} < 0$. Hence, Theorem 8.3.1 shows that the solution we have found is optimal.

8.3.4 (a) With $F(t, y, \dot{y}) = \ln\big[y - \sigma\dot{y} - \bar{z}l(t)\big]$ we get $F'_y = 1/[y - \sigma\dot{y} - \bar{z}l(t)]$ and $F'_{\dot{y}} = -\sigma/[y - \sigma\dot{y} - \bar{z}l(t)]$. The Euler equation is then

$$\frac{1}{y - \sigma\dot{y} - \bar{z}l(t)} - \frac{d}{dt}\Big[\frac{-\sigma}{y - \sigma\dot{y} - \bar{z}l(t)}\Big] = 0 \iff \frac{1}{y - \sigma\dot{y} - \bar{z}l(t)} - \frac{\sigma(\dot{y} - \sigma\ddot{y} - \bar{z}\dot{l}(t))}{[y - \sigma\dot{y} - \bar{z}l(t)]^2} = 0$$

$$\iff y - \sigma\dot{y} - \bar{z}l(t) - \sigma\dot{y} + \sigma^2\ddot{y} + \sigma\bar{z}\dot{l}(t) = 0 \iff \ddot{y} - \frac{2}{\sigma}\dot{y} + \frac{1}{\sigma^2}y = \frac{\bar{z}}{\sigma^2}\big[l(t) - \sigma\dot{l}(t)\big]$$

(b) With $l(t) = l_0e^{\alpha t}$ the Euler equation becomes

$$\ddot{y} - \frac{2}{\sigma}\dot{y} + \frac{1}{\sigma^2}y = \frac{\bar{z}(1 - \alpha\sigma)l_0}{\sigma^2}e^{\alpha t} \tag{$*$}$$

The characteristic equation $r^2 - (2/\sigma)r + 1/\sigma^2 = 0$ has the double root $r_1 = r_2 = 1/\sigma$, so the corresponding homogeneous differential equation has the general solution $y_{\mathrm{H}} = Ae^{t/\sigma} + Bte^{t/\sigma}$. To find a particular integral of $(*)$ we try with a function $u^* = Ce^{\alpha t}$. We get

$$\ddot{u}^* - \frac{2}{\sigma}\dot{u}^* + \frac{1}{\sigma^2}u^* = \alpha^2Ce^{\alpha t} - \frac{2\alpha}{\sigma}Ce^{\alpha t} + \frac{1}{\sigma^2}Ce^{\alpha t} = C\frac{(1 - \alpha\sigma)^2}{\sigma^2}e^{\alpha t}$$

It follows that u^* is a solution of $(*)$ if and only if $C = \bar{z}l_0/(1 - \alpha\sigma)$, and the general solution of $(*)$ is

$$y = y_{\mathrm{H}} + u^* = Ae^{t/\sigma} + Bte^{t/\sigma} + \frac{\bar{z}l_0}{1 - \alpha\sigma}e^{\alpha t}$$

(If $\alpha\sigma = 1$, then $(*)$ is homogeneous and the general solution is y_{H}.)

8.4

8.4.1 With $F(t, K, \dot{K}) = e^{-t/4}\ln(2K - \dot{K})$ we get $F'_K = 2e^{-t/4}/(2K - \dot{K})$ and $F'_{\dot{K}} = -e^{-t/4}/(2K - \dot{K})$. The Euler equation is

$$e^{-t/4}\frac{2}{2K - \dot{K}} - \frac{d}{dt}\Big(-e^{-t/4}\frac{1}{2K - \dot{K}}\Big) = 0 \iff \frac{2e^{-t/4}}{2K - \dot{K}} - \frac{e^{-t/4}}{4(2K - \dot{K})} - \frac{e^{-t/4}(2\dot{K} - \ddot{K})}{(2K - \dot{K})^2} = 0$$

$$\iff \frac{e^{-t/4}}{4(2K - \dot{K})^2}\big[8(2K - \dot{K}) - (2K - \dot{K}) - 4(2\dot{K} - \ddot{K})\big] = 0$$

$$\iff 4\ddot{K} - 15\dot{K} + 14K = 0$$

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The characteristic equation of the Euler equation is $4r^2 - 15r + 14 = 0$, which has the roots $r_1 = 2$ and $r_2 = 7/4$. Hence the general solution of the Euler equation is $K = Ae^{2t} + Be^{7t/4}$. The boundary conditions yield the equations

$$K(0) = A + B = K_0, \qquad K(T) = Ae^{2T} + Be^{7T/4} = K_T$$

By means of Cramer's rule or by other methods you will find that

$$A = \frac{K_T - e^{7T/4}K_0}{e^{2T} - e^{7T/4}}, \qquad B = \frac{e^{2T}K_0 - K_T}{e^{2T} - e^{7T/4}}$$

8.4.2 (a) Let $F(t, x, \dot{x}) = \left(\frac{1}{100}tx - \dot{x}^2\right)e^{-t/10}$. Then $F'_x = te^{-t/10}/100$ and $F'_{\dot{x}} = -2\dot{x}e^{-t/10}$, and the Euler equation becomes

$$\frac{t}{100}e^{-t/10} - \frac{d}{dt}\left(-2\dot{x}e^{-t/10}\right) = 0 \iff \frac{t}{100}e^{-t/10} + 2\ddot{x}e^{-t/10} - \frac{2}{10}\dot{x}e^{-t/10} = 0$$
$$\iff \ddot{x} - \frac{1}{10}\dot{x} = -\frac{1}{200}t \qquad (*)$$

The general solution of the corresponding homogeneous equation is

$$x_{\mathrm{H}} = A + Be^{t/10}$$

To find a particular solution u^* of $(*)$ we try with $u = Pt^2 + Qt$. Then $\dot{u} = 2Pt + Q$ and $\ddot{u} = 2P$, and if we insert this into $(*)$ we get

$$2P - \frac{P}{5}t - \frac{Q}{10} = -\frac{1}{200}t$$

This yields $P = 5/200 = 1/40$ and $Q = 20P = 1/2$, so the general solution of the Euler equation is

$$x = A + Be^{t/10} + \frac{1}{40}t^2 + \frac{1}{2}t$$

The boundary conditions $x(0) = 0$ and $x(T) = S$ yield the equations

$$A + B = 0, \qquad A + Be^{T/10} + T^2/40 + T/2 = S$$

with the solution

$$A = -B = \frac{T^2/40 + T/2 - S}{e^{T/10} - 1}$$

Since $F(t, x, \dot{x})$ is concave with respect to $(x, \dot{x})$, this is indeed an optimal solution.

(b) With $T = 10$ and $S = 20$ we get $B = -5/2 + 5 - 20/(e - 1) = 25/2(e - 1)$, and the optimal solution is

$$x = \frac{25(e^{t/10} - 1)}{2(e - 1)} + \frac{1}{40}t^2 + \frac{1}{2}t$$

8.4.3 With $F(t, K, \dot{K}) = U(C, t)$ we get $F'_K = U'_C C'_K = U'_C(f'_k - \delta)$ and $F'_{\dot{K}} = U'_C \cdot (-1) = -U'_C$. The Euler equation is

$$U'_C(f'_K - \delta) + \frac{d}{dt}U'_C = 0 \iff U'_C(f'_K - \delta) + U''_{CC}\dot{C} + U''_{Ct} = 0$$

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The Euler equation implies $\dot{C} = -[U''_{Ct} + U'_C(f'_K - \delta)]/U''_{CC}$, and therefore

$$\frac{\dot{C}}{C} = -\frac{U''_{Ct} + U'_C(f'_K - \delta)}{CU''_{CC}} = -\frac{1}{\check{\omega}}\Big(\frac{U''_{Ct}}{U'_C} + f'_K - \delta\Big)$$

where $\check{\omega} = CU''_{CC}/U'_C$ is the elasticity of marginal utility with respect to consumption.

8.4.4 (a) $F(t, p, \dot{p}) = pD(p, \dot{p}) - b(D(p, \dot{p}))$ yields $F'_p = D + pD'_p - b'(D)D'_p = D + [p - b'(D)]D'_p$ and $F'_{\dot{p}} = [p - b'(D)]D'_{\dot{p}}$, so the Euler equation is

$$F'_p - \frac{d}{dt}(F'_{\dot{p}}) = 0 \iff D + \big[p - b'(D)\big]D'_p - \frac{d}{dt}\big[[p - b'(D)]D'_{\dot{p}}\big] = 0$$

(b) With $b(x) = \alpha x^2 + \beta x + \gamma$ and $x = D(p, \dot{p}) = Ap + B\dot{p} + C$, we get $b'(x) = 2\alpha x + \beta$, $\partial D/\partial p = A$, and $\partial D/\partial \dot{p} = B$. Insertion into the Euler equation gives

$$Ap + B\dot{p} + C + [p - 2\alpha(Ap + B\dot{p} + C) - \beta]A - \frac{d}{dt}\big[\big(p - 2\alpha(Ap + B\dot{p} + C) - \beta\big)B\big] = 0$$

which yields $Ap + B\dot{p} + C + Ap - 2\alpha A^2 p - 2\alpha AB\dot{p} - 2\alpha AC - \beta A - B\dot{p} + 2\alpha AB\dot{p} + 2\alpha B^2\ddot{p} = 0$. Rearranging the terms we can write the Euler equation as

$$\ddot{p} - \frac{A^2 - A/\alpha}{B^2}p = \frac{\beta A + 2\alpha AC - C}{2\alpha B^2}$$

and it is easy to see that we get the answer given in the book.

8.5

8.5.2 (a) The Euler equation is

$$(-2\dot{x} - 10x)e^{-t} - \frac{d}{dt}[(-2\dot{x} - 2x)e^{-t}] = 0 \iff e^{-t}(-2\dot{x} - 10x + 2\ddot{x} + 2\dot{x} - 2\dot{x} - 2x) = 0$$
$$\iff \ddot{x} + \dot{x} - 6x = 0$$

The general solution of this equation is $x = Ae^{3t} + Be^{-2t}$, and the boundary conditions yield the equations

$$x(0) = A + B = 0, \qquad x(1) = Ae^3 + Be^{-2} = 1$$

which have the solution $A = -B = 1/(e^3 - e^{-2})$. The integrand $F(t, x, \dot{x}) = (10 - \dot{x}^2 - 2x\dot{x} - 5x^2)e^{-t}$ is concave with respect to $(x, \dot{x})$ (look at the Hessian or note that $-\dot{x}^2 - 2x\dot{x} - 5x^2 = -(\dot{x} - x)^2 - 4x^2$), so the solution we have found is optimal (Theorem 8.5.1).

(b) (i) With $x(1)$ free, we get the transversality condition $\big[F'_{\dot{x}}\big]_{t=1} = 0$, which implies

$$\dot{x}(1) + x(1) = 0 \iff 3Ae^3 - 2Be^{-2} + Ae^3 + Be^{-2} = 0 \iff 4Ae^3 - Be^{-2} = 0$$

Together with the equation $A + B = x(0) = 0$, this yields $A = B = 0$, so the optimal solution is $x \equiv 0$.

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(ii) With the terminal condition $x(1) \geq 2$ the transversality condition is $\left[F'_{\dot{x}}\right]_{t=1} \leq 0$, with $\left[F'_{\dot{x}}\right]_{t=1} = 0$ if $x(1) > 2$. In our case this reduces to

$$-\dot{x}(1) - x(1) \leq 0, \quad \text{and} \quad -\dot{x}(1) - x(1) = 0 \text{ if } x(1) > 2$$

Since $x(t) = Ae^{3t} - Ae^{-2t}$, we get

$$-A(4e^3 + e^{-2}) \leq 0, \quad \text{and} \quad -A(4e^3 + e^{-2}) = 0 \text{ if } x(1) = A(e^3 - e^{-2}) > 2$$

From the last condition it is clear that we cannot have $A(e^3 - e^{-2}) > 2$, for then we must have $A = 0$, and that is impossible. Therefore $A(e^3 - e^{-2}) = 2$ and $x = \dfrac{2(e^{3t} - e^{-2})}{e^3 - e^{-2}}$.

8.5.4 With $U(C) = a - e^{-bC}$, we get $U'(C) = be^{-bC}$ and $U''(C) = -b^2e^{-bC}$, so equation (∗) in Example 8.5.3 reduces to $\ddot{A} - r\dot{A} = (\rho - r)/b$. Putting $z = \dot{A}$, we get $\dot{z} - rz = (\rho - r)/b$. This is a linear first-order differential equation with constant coefficients, and the solution is $z = Pe^{rt} - (\rho - r)/br$. Then $A = \int z\,dt = Ke^{rt} - (\rho - r)t/br + L$, where $K = P/r$ and L are constants. These constants are determined by the boundary conditions $A(0) = A_0$ and $A(T) = A_t$.

8.5.5 (a) The conditions are $-(d/dt)[C'_{\dot{x}}(t, \dot{x})e^{-rt}] = 0$ and $C'_{\dot{x}}(5, \dot{x}(5)) \geq 0$ $(= 0$ if $x(5) > 1500)$. It follows that $C'_{\dot{x}}(t, \dot{x}) = Ke^{-rt}$, where K is a constant.

(b) It follows from part (a) that if $r = 0$ then $C'_{\dot{x}}(t, \dot{x}) = K$ must be constant. With $C(t, u) = g(u)$ it means that $g'(\dot{x})$ must be constant. Since g' is strictly increasing, $\dot{x}$ must also be a constant, so $x = At$. We must have $5A \geq 1500$. Since $C'_{\dot{x}}(t, \dot{x}) = g'(\dot{x}) = g'(A) > 0$, the transversality condition shows that $x(5) = 1500$, so $A = 300$ and $x = 300t$. Thus planting will take place at the constant rate of 300 hectares per year.

9 Control Theory: Basic Techniques

9.2

9.2.3 We transform the problem to a maximization problem by maximizing $\int_0^1 [-x(t) - u(t)^2]\,dt$. The Hamiltonian $H(t, x, u, p) = -x - u^2 - pu$ is concave in (x, u), so according to Theorem 9.2.1, the following conditions are sufficient for optimality (using (9.2.7)):

(i) $H'_u(t, x^*(t), u^*(t), p(t)) = -2u^*(t) - p(t) = 0$;
(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 1$, $p(1) = 0$;
(iii) $\dot{x}^*(t) = -u^*(t)$, $x^*(0) = 0$.

From (ii), $p(t) = t + A$ for some constant A, and $p(1) = 0$ gives $A = -1$ and so $p(t) = t - 1$. From (i) we have $u^*(t) = -\frac{1}{2}p(t) = \frac{1}{2}(1 - t)$. It remains to determine $x^*(t)$. From (iii), $\dot{x}^*(t) = -u^*(t) = \frac{1}{2}t - \frac{1}{2}$, and so $x^*(t) = \frac{1}{4}t^2 - \frac{1}{2}t + B$. With $x^*(0) = 0$, we get $B = 0$. So the optimal solution is $u^*(t) = \frac{1}{2}(1 - t)$, $x^*(t) = \frac{1}{4}t^2 - \frac{1}{2}t$, with $p(t) = t - 1$.

9.2.4 The Hamiltonian $H(t, x, u, p) = 1 - 4x - 2u^2 + pu$ is concave in (x, u), so according to Theorem 9.2.1 (using (9.2.7), the following conditions are sufficient for optimality:

(i) $H'_u(t, x^*(t), u^*(t), p(t)) = -4u^*(t) + p(t) = 0$;
(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 4$, $p(10) = 0$;
(iii) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 0$.

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From (ii), $p(t) = 4t + A$ for some constant A and $p(10) = 40 + A = 0$, so $A = -40$ and $p(t) = 4t - 40$. From (i) we have $u^*(t) = \frac{1}{4}(4t - 40) = t - 10$. From (iii), $\dot{x}^*(t) = u^*(t) = t - 10$, and with $x^*(0) = 0$, we get $x^*(t) = \frac{1}{2}t^2 - 10t$.

9.2.5 The Hamiltonian $H(t, x, u, p) = x - u^2 + p(x + u)$ is concave in (x, u), so according to Theorem 9.2.1 (using (9.2.7), the following conditions are sufficient for optimality:

(i) $H'_u(t, x^*(t), u^*(t), p(t)) = -2u^*(t) + p(t) = 0$;
(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -1 - p(t)$, $p(T) = 0$;
(iii) $\dot{x}^*(t) = x^*(t) + u^*(t)$, $x^*(0) = 0$.

From (ii) we get the linear differential equation $\dot{p}(t) + p(t) = -1$. The general solution is $p(t) = Ae^{-t} - 1$, and $p(T) = 0$ gives $A = e^T$, so $p(t) = e^{T-t} - 1$. From (i) we have $u^*(t) = \frac{1}{2}(e^{T-t} - 1)$. Finally, from (iii), $\dot{x}^*(t) = x^*(t) + \frac{1}{2}(e^{T-t} - 1)$, with general solution $x^*(t) = Be^t + \frac{1}{2}e^t \int e^{-t}(e^{T-t} - 1)\,dt = Be^t - \frac{1}{4}e^{T-t} + \frac{1}{2}$. The condition $x^*(0) = 0$ gives $B = \frac{1}{4}e^T - \frac{1}{2}$. Thus, $x^*(t) = \frac{1}{4}e^{T+t} - \frac{1}{4}e^{T-t} - \frac{1}{2}e^t + \frac{1}{2}$.

9.2.6 (b) Conditions (7) and (5) reduce to

(i) $I^*(t) = \frac{1}{2}p(t)$;
(ii) $\dot{p}(t) - 0.1p(t) = -1 + 0.06K^*(t)$.

Moreover, $K^*(t) = I^*(t) - 0.1K^*(t) = \frac{1}{2}p(t) - 0.1K^*(t)$. Thus $(K^*(t), p(t))$ must satisfy

(iii) $\dot{K} = \frac{1}{2}p - 0.1K$, and
(iv) $\dot{p} - 0.1p = -1 + 0.06K$.

From (iii) we get $p = 2\dot{K} + 0.2K$, and then $\dot{p} = 2\ddot{K} + 0.2\dot{K}$. Inserting these results into (iv) and simplifying yields $\ddot{K} - 0.04K = -0.5$.

9.4

9.4.2 The Hamiltonian $H(t, x, p, u) = 1 - x^2 - u^2 + pu$ is concave in (x, u), so according to Theorem 9.4.2 (using (9.2.7), the following conditions are sufficient for optimality:

(i) $H'_u(t, x^*(t), u^*(t), p(t)) = -2u^*(t) + p(t) = 0$;
(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 2x^*(t)$;
(iii) $p(1) \geq 0$, and $p(1) = 0$ if $x^*(1) > 1$;
(iv) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 0$, $x^*(1) \geq 1$.

From $\dot{p}(t) = 2x^*(t)$ we get $\ddot{p}(t) = 2\dot{x}^*(t) = 2u^*(t) = p(t)$. It follows that $p(t) = Ae^t + Be^{-t}$ for suitable constants A and B. Furthermore, $x^*(t) = \frac{1}{2}\dot{p} = \frac{1}{2}(Ae^t - Be^{-t})$, and since $x^*(0) = 0$ gives $B = A$, we have $x^*(t) = \frac{1}{2}A(e^t - e^{-t})$, $u^*(t) = \dot{x}^*(t) = \frac{1}{2}A(e^t + e^{-t})$, and $p(t) = A(e^t + e^{-t})$. If $x^*(1) > 1$, we must have $p(1) = 0$, and therefore $A = 0$, but that would give $x^*(t) = 0$ for all t, which contradicts $x^*(1) \geq 1$. Therefore we must have $x^*(1) = 1$, which gives $A = 2/(e - e^{-1}) = 2e/(e^2 - 1)$. (The value of A here is twice the value of A given in the book.)

9.4.3 (a) As in Example 9.2.1, we have $p(t) = -\frac{1}{2}(T^2 - t^2)$, but this time $u^*(t)$ is the value of u that maximizes $H(t, x^*(t), u, p(t)) = 1 - tx^*(t) - u^2 + p(t)u$ for $u \in [0, 1]$. Note that $H'_u = -2u + p(t) = -2u - \frac{1}{2}(T^2 - t^2) < 0$ for all $t < T$. Thus the optimal choice of u must be $u^*(t) \equiv 0$, and then $x^*(t) \equiv x_0$.

(b) Also in this case $H'_u = -2u + p(t) = -2u - \frac{1}{2}(T^2 - t^2)$ and $H''_{uu} = -2$. So for each t in $[0, T]$, the optimal $u^*(t)$ must maximize the concave function $H(t, x^*(t), u, p(t))$ for $u \in [-1, 1]$. Note that $H'_u = 0$ when $u = -\frac{1}{4}(T^2 - t^2)$, and this nonpositive number is an interior point of $[-1, 1]$ provided

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$-\frac{1}{4}(T^2 - t^2) > -1$, i.e. $t > \sqrt{T^2 - 4}$. If $t \le \sqrt{T^2 - 4}$, then $u = -1$ is optimal. (Choosing $u^*(t) = 1$ cannot be optimal because H'_u is negative for $u = 1$.) For the rest see the answer in the book.

9.4.4 (a) Since $H(t, x, p, u) = x + pu$ is concave in (x, u), the following conditions are sufficient for optimality:

(i) $u = u^*(t)$ maximizes $p(t)u$ for $u \in [0, 1]$;
(ii) $\dot{p} = -1$;
(iii) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 0$, $x^*(10) = 2$.

From (ii) we get $p(t) = A - t$ for some constant A, and (i) implies that $u^*(t) = 1$ if $p(t) > 0$ and $u^*(t) = 0$ if $p(t) < 0$. Now $u^*(t) \equiv 0$ and $u^*(t) \equiv 1$ are both impossible because $\dot{x}^*(t) = u^*(t)$ and $x^*(0) = 0$ contradict $x^*(10) = 2$. Since $p(t)$ is strictly decreasing, we conclude that for some t^* in $(0, 10)$ we have $p(t) > 0$, and thus $u^*(t) = 1$, for t in $[0, t^*]$, and $p(t) < 0$ with $u^*(t) = 0$ for t in $(t^*, 10]$. At t^* we have $p(t^*) = 0$, so $A = t^*$ and $p(t) = t^* - t$. We see that $x^*(t) = t$ for t in $[0, t^*]$ and $x^*(t) = t^*$ for t in $(t^*, 10]$. Since $x^*(10) = 2$, we conclude that $t^* = 2$, and the solution is as given in the book.

(b) As in (a) we find that for some t^* in $(0, T)$, $u^*(t) = 1$ in $[0, t^*]$ and $u^*(t) = 0$ for t in $(t^*, T]$. Then $x^*(t) = t + x_0$ for t in $[0, t^*]$ and $x^*(t) = t^* + x_0$ for t in $(t^*, T]$. Since $x^*(T) = x_1$, we conclude that $t^* = x_1 - x_0$, and the solution is as given in the book.

9.4.6 The Hamiltonian $H(t, x, u, p) = [10u - u^2 - 2]e^{-0.1t} - pu$ is concave in (x, u), so according to Theorem 9.4.2, the following conditions are sufficient for optimality:

(i) $u^*(t)$ maximizes $H(t, x^*(t), u, p(t)) = [10u - u^2 - 2]e^{-0.1t} - p(t)u$ for $u \ge 0$;
(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 0$;
(iii) $p(5) \ge 0$, and $p(5) = 0$ if $x^*(5) > 0$;
(iv) $\dot{x}^*(t) = -u^*(t)$, $x^*(0) = 10$, $x^*(5) \ge 0$.

From (ii) we have $p(t) \equiv \bar{p}$ for some constant $\bar{p}$. Suppose $u^*(t) > 0$ for all t in $[0, 5]$. Then (i) is equivalent to $H'_u(t, x^*(t), u^*(t), p(t)) = 0$, that is,

(v) $[10 - 2u^*(t)]e^{-0.1t} = \bar{p}$.

If $\bar{p} = 0$, then (v) implies that $u^*(t) \equiv 5$, and then $\dot{x}^*(t) = -5$. With $x^*(0) = 10$, we get $x^*(t) = 10 - 5t$, which gives $x^*(5) = -15$, a contradiction. Hence $\bar{p} > 0$ and from (iii) (and $x^*(5) \ge 0$) we conclude that $x^*(5) = 0$. From (v) we have $u^*(t) = 5 - \frac{1}{2}\bar{p}e^{0.1t}$, and then $\dot{x}^*(t) = -u^*(t)$ with $x^*(0) = 10$ gives $x^*(t) = 5\bar{p}(e^{0.1t} - 1) - 5t + 10$. The condition $x^*(5) = 0$ gives $\bar{p} = 3/(e^{0.5} - 1)$. Since all the conditions (i)–(iv) are satisfied by the pair $(x^*(t), u^*(t))$, we have found an optimal solution.

9.4.7 (b) The Hamiltonian $H(t, x, u, p) = -(ax + bu^2) + pu$ is concave in (x, u). To check the maximum condition (9.4.5) in Theorem 9.4.1, it suffices to check that, if $u^*(t) > 0$, then $(H'_u)^* = -2bu^*(t) + p(t) = 0$, and if $u^*(t) = 0$, then $(H'_u)^* = -2bu^*(t) + p(t) \le 0$. Also $p(t) = at + A$ for some constant A.

Suppose $u^*(t) > 0$ for all t in $[0, T]$. Then $u^*(t) = p(t)/2b = (1/2b)(at + A)$. Moreover, $\dot{x}^*(t) = u^*(t) = (1/2b)(at + A)$, so $x^*(t) = (1/2b)(\frac{1}{2}at^2 + At) + C$. Here $x^*(0) = 0$ yields $C = 0$, and $x^*(T) = B$ gives $A = 2bB/T - aT/2$. Thus

$$u^*(t) = a(2t - T)/4b + B/T \quad \text{and} \quad x^*(t) = at(t - T)/4b + Bt/T$$

Note that $u^*(t)$ is increasing in t, and $u^*(0) = B/T - aT/4b \ge 0$ if and only if $B \ge aT^2/4b$. So in this case we have found an optimal solution, because it is easy to check that all the conditions in Theorem

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9.4.2 are satisfied. (This solution is valid if the required total production B is large relative to the time period T available, and the storage cost a is sufficiently small relative to the unit production cost b. See Kamien and Schwartz (1991) for further economic interpretations.)

Suppose $B < aT^2/4b$. Then we guess that production is postponed in an initial time period. We cannot have $u^*(t) \equiv 0$ because then $x^*(t) \equiv 0$, which contradicts $x^*(T) = B > 0$ (assumed implicitly). If $u^*(t^{*+}) > 0$, then by continuity of $p(t)$, the unique maximizer of H would be > 0 for some $t < t^*$, with t close to t^*. So $u^*(t^{*+}) = 0$, and $u^*(t) = x^*(t) \equiv 0$ in $[0, t^*]$. In particular, $x^*(t^*) = 0$. In $(t^*, T]$ we have as before $u^*(t) = (1/2b)(at+A)$, and since $u^*(t^{*+}) = 0$, we have $A = -at^*$, and $u^*(t) = (a/2b)(t-t^*)$. Then $\dot{x}^*(t) = u^*(t) = (a/2b)(t-t^*)$, and with $x^*(t^*) = 0$, $x^*(t) = (a/4b)(t-t^*)^2$. Now t^* is determined by the condition $x^*(T) = B$, which gives $(a/4b)(T - t^*)^2 = B$, or $T - t^* = \pm\sqrt{4bB/a}$. The minus sign would make $t^* > T$, so we must have $t^* = T - 2\sqrt{bB/a}$. Note that $t^* > 0 \iff B < aT^2/4b$. We end up with the following suggestion for an optimal solution:

$$u^*(t) = \begin{cases} 0 & \text{if } t \in [0, t^*] \\ \dfrac{a(t-t^*)}{2b} & \text{if } t \in (t^*, T] \end{cases}, \qquad x^*(t) = \begin{cases} 0 & \text{if } t \in [0, t^*] \\ \dfrac{a(t-t^*)^2}{4b} & \text{if } t \in (t^*, T] \end{cases}, \qquad p(t) = a(t-t^*)$$

with $t^* = T - 2\sqrt{bB/a}$.

It remains to check that the proposed solution satisfies all the conditions in Theorem 9.4.2. The pair (x^*, u^*) is admissible and satisfies the terminal condition $x^*(T) = B$, and $\dot{p} = a = -(H'_x)^*$. It remains only to check the maximum condition. For t in $(t^*, T]$ we have $u^*(t) > 0$ and we see that $(H'_u)^* = -2bu^*(t) + p(t) = 0$. For t in $[0, t^*]$, we have $u^*(t) = 0$ and $(H'_u)^* = -2bu^*(t) + p(t) = a(t - t^*) \le 0$, as it should be. So we have found an optimal solution.

Note: When we use sufficient conditions, we can use "wild" guesses about the optimal control and the optimal path, as long as we really check that all the conditions in the sufficiency theorem are satisfied.

9.4.8 The Hamiltonian is $H(t, x, p, u) = x^2 - 2u + pu = x^2 + (p-2)u$. (Since $x(2)$ is free, we put $p_0 = 1$.) The maximum principle (Theorem 9.4.1) gives the following necessary conditions:

(i) $u = u^*(t)$ maximizes $(p(t) - 2)u$ for $u \in [0, 1]$;
(ii) $\dot{p} = -H'_x(t, x^*(t), u^*(t), p(t)) = -2x^*(t)$, $p(2) = 0$;
(iii) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 1$.

From (i) we see that $p(t) > 2 \Rightarrow u^*(t) = 1$ and $p(t) < 2 \Rightarrow u^*(t) = 0$. The function x^* must be increasing, because $\dot{x}^*(t) \ge 0$. Hence $\dot{p}(t) = -2x^*(t) \le -2x^*(0) = -2 < 0$. Consequently p is strictly decreasing. Since $p(2) = 0$, we have $p(t) > 0$ for t in $[0, 2)$. Because $\dot{p} \le -2$, we see that $p(2) - p(0) = \int_0^2 \dot{p}(t)\,dt \le \int_0^2 (-2)\,dt = -4$, so $p(0) \ge p(2) + 4 = 4$. There is therefore a unique t^* in $(0, 2)$ with $p(t^*) = 2$, and

$$p(t) \begin{cases} > 2 & \text{for } t \in [0, t^*) \\ < 2 & \text{for } t \in (t^*, 2] \end{cases} \implies u^*(t) = \begin{cases} 1 & \text{for } t \in [0, t^*] \\ 0 & \text{for } t \in (t^*, 2] \end{cases}$$

For $t \le t^*$, we have $\dot{x}^*(t) = u^*(t) = 1$, so $x^*(t) = x^*(0) + t = 1 + t$. Moreover, $\dot{p}(t) = -2x(t) = -2 - 2t$, which gives $p(t) = -2t - t^2 + C_0$ for a suitable constant C_0. Since x^* is continuous, $x^*(t^*) = 1 + t^*$ (the limit of $x^*(t)$ as t approaches t^* from the left).

For $t \ge t^*$, we have $\dot{x}^*(t) = u^*(t) = 0$, so $x^*(t)$ remains constant, $x^*(t) = x^*(t^*) = 1 + t^*$, and $\dot{p}(t) = -2x^*(t) = -2 - 2t^*$, so $p(t) = -2(1+t^*)t + C_1$, where $C_1 = p(2) + 2(1+t^*)2 = 4(1+t^*)$. Now, $p(t^*)$ must equal 2, so $-2(1+t^*)t^* + 4(1+t^*) = 2$, or $(t^*)^2 - t^* - 1 = 0$. This quadratic equation

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gives $t^* = (1 \pm \sqrt{5})/2$, and since t^* must be positive, $t^* = (1+\sqrt{5})/2$. We know that for $t \geq t^*$ we have $p(t) = -2(1+t^*)t + C_1 = -2(1+t^*)t + 4(1+t^*) = 2(1+t^*)(2-t) = (3+2\sqrt{5})(2-t)$. We also know that there is a constant C_0 such that $p(t) = -2t - t^2 + C_0$ for $t \leq t^*$. To determine the constant C_0, we use the equation $p(t^*) = 2$, which gives $C_0 - 2t^* - (t^*)^2 = 2$, or $C_0 = 2 + 2t^* + (t^*)^2 = \frac{9}{2} + \frac{3}{2}\sqrt{5}$. The only possible solution is spelled out in the answer in the book.

9.5

9.5.1 With $F(t, x, \dot{x}) = 2xe^{-t} - 2x\dot{x} - \dot{x}^2$ we get $F'_x = 2e^{-t} - 2\dot{x}$ and $F'_{\dot{x}} = -2x - 2\dot{x}$. The Euler equation is then $2e^{-t} - 2\dot{x} - \frac{d}{dt}(-2x - 2\dot{x}) = 0$, or $2e^{-t} - 2\dot{x} + 2\dot{x} + 2\ddot{x} = 0$, which reduces to $\ddot{x} = -e^{-t}$. It follows that $\dot{x} = e^{-t} + A$ and $x = -e^{-t} + At + B$ for suitable constants A and B. From the boundary conditions $x(0) = 0$ and $x(1) = 1$ we get $A = e^{-1}$ and $B = 1$, so the only admissible solution of the Euler equation is $x^*(t) = -e^{-t} + e^{-1}t + 1$.

The associated control problem is

$$\max \int_0^1 (2xe^{-t} - 2xu - u^2)\, dt, \quad \dot{x} = u, \quad x(0) = 0, \quad x(1) = 1, \quad u \in (-\infty, \infty)$$

The Hamiltonian (with $p_0 = 1$) is then $H(t, x, u, p) = 2xe^{-t} - 2xu - u^2 + pu$. (Incidentally, if $p_0 = 0$, then the Hamiltonian would simply be pu. If $p(t)u$ has a maximum for some u in $(-\infty, \infty)$, then $p(t) = 0$, but $p(t) = 0$ is impossible when $p_0 = 0$.) We have $H'_u = -2x - 2u + p$ and $H'_x = 2e^{-t} - 2u$. Since the control region is open, $H'_u(t, x^*(t), u^*(t), p(t)) = 0$, so (i) $u^*(t) = \frac{1}{2}p(t) - x^*(t)$. Moreover (ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -2e^{-t} + 2u^*(t) = -2e^{-t} + 2\dot{x}^*(t)$. From $\dot{x}^*(t) = u^*(t)$ we get $\ddot{x}^*(t) = \dot{u}^*(t) = \frac{1}{2}\dot{p} - \dot{x}^*(t) = -e^{-t}$, using (i) and (ii). Hence $x^*(t)$ must be a solution of the Euler equation that we found in (a). As we found in (a), there is precisely one solution that also satisfies the boundary conditions on $x^*(t)$. Now that we know $x^*(t)$, the control function $u^*(t)$ and the adjoint function $p(t)$ are given by the equations above.

9.5.2 With $F(t, x, \dot{x}) = 3 - x^2 - 2\dot{x}^2$, $F'_x = -2x$ and $F'_{\dot{x}} = -4\dot{x}$, so the Euler equation $F'_x - (d/dt)F'_{\dot{x}} = 0$ reduces to $\ddot{x} - \frac{1}{2}x = 0$. The characteristic equation $r^2 = \frac{1}{2}$ has solutions $r_1 = \frac{1}{2}\sqrt{2}$ and $r_2 = -\frac{1}{2}\sqrt{2}$. So if $x^*(t)$ solves the problem, then we must have $x^*(t) = Ae^{\frac{1}{2}\sqrt{2}t} + Be^{-\frac{1}{2}\sqrt{2}t}$.

Since $x^*(0) = 1$, we have $A + B = 1$, or $B = 1 - A$, and, moreover, $x^*(2) = Ae^{\sqrt{2}} + (1-A)e^{-\sqrt{2}} \geq 4$ requires

$$A \geq (4 - e^{-\sqrt{2}})/(e^{\sqrt{2}} - e^{-\sqrt{2}}) = (4e^{\sqrt{2}} - 1)/(e^{2\sqrt{2}} - 1) \approx 0.97 \tag{$*$}$$

We now invoke the transversality condition (8.5.3):

$$(F'_{\dot{x}})^*_{t=2} = -4\dot{x}^*(2) \leq 0, \quad \text{with} \quad -4\dot{x}^*(2) = 0 \quad \text{if} \quad x^*(2) > 4$$

Equivalently, $\dot{x}^*(2) \geq 0$, with $\dot{x}^*(2) = 0$ if $x^*(2) > 4$. Since $\dot{x}^*(t) = \frac{1}{2}\sqrt{2}[Ae^{\frac{1}{2}\sqrt{2}t} - (1-A)e^{-\frac{1}{2}\sqrt{2}t}]$, we have $\dot{x}^*(2) = \frac{1}{2}\sqrt{2}[Ae^{\sqrt{2}} - (1-A)e^{-\sqrt{2}}] = 0$ provided $A = e^{-\sqrt{2}}/(e^{\sqrt{2}} + e^{-\sqrt{2}}) \approx 0.06$, contradicting ($*$). We conclude that $x^*(2) = 4$ and thus $A = (4e^{\sqrt{2}} - 1)/(e^{2\sqrt{2}} - 1)$. The function F is (for t fixed) concave in $(x, \dot{x})$, so we have found the solution.

The control problem is

$$\max \int_0^2 (3 - x^2 - 2u^2)\, dt, \quad \dot{x} = u, \quad x(0) = 1, \quad x(2) \geq 4$$

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The Hamiltonian is $H = 3 - x^2 - 2u^2 + pu$, so $H'_x = -2x$ and $H'_u = -4u + p$. If $(x^*(t), u^*(t))$ is optimal then (i) $u^*(t) = \frac{1}{4}p(t)$; (ii) $\dot{p}(t) = 2x^*(t)$; (iii) $\dot{x}^*(t) = u^*(t)$. Differentiating (iii) yields $\ddot{x}^*(t) = \dot{u}^*(t) = \frac{1}{4}\dot{p}(t) = \frac{1}{4}2x^*(t) = \frac{1}{2}x^*(t)$, which is the same differential equation as before. The rest is easy.

9.5.3 With $F(t, x, \dot{x}) = (-2\dot{x} - \dot{x}^2)e^{-t/10}$, $F'_x = 0$ and $F'_{\dot{x}} = (-2 - \dot{x})e^{-t/10}$, so the Euler equation is $-\dfrac{d}{dt}(-2 - \dot{x})e^{-t/10} = 0$. See the answer in the book. The function F is (for t fixed) concave in $(x, \dot{x})$, so we have found the solution.

The Hamiltonian for the control problem is $H = (-2u - u^2)e^{-t/10} + pu$. Here $\dot{p}(t) = -(H'_x)^* = 0$, so $p(t) = \bar{p}$, a constant. Moreover, $(H'_u)^* = (-2 - 2u^*(t))e^{-t/10} + p(t) = 0$, and thus $u^*(t) = \frac{1}{2}\bar{p}e^{t/10} - 1$. Since $\dot{x}^*(t) = u^*(t)$, integration gives $x^*(t) = 5\bar{p}e^{t/10} - t + A$. The boundary conditions yield $\bar{p} = 0$ and $A = 1$, so we get the same solution.

9.6

9.6.1 (a) The Hamiltonian $H(t, x, u, p) = x - \frac{1}{2}u^2 + pu$ is concave in (x, u), so according to Theorem 9.2.2 (see (9.2.7)), the following conditions are sufficient for optimality:

(i) $H'_u(t, x^*(t), u^*(t), p(t)) = -u^*(t) + p(t) = 0$;

(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -1$ with $p(T) = 0$;

(iii) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = x_0$.

From (i) and (ii) we have $u^*(t) = p(t) = T - t$. Since $\dot{x}^*(t) = u^*(t) = T - t$ and $x^*(0) = x_0$, we get $x^*(t) = Tt - \frac{1}{2}t^2 + x_0$.

(b) The optimal value function is

$$V(x_0, T) = \int_0^T \big(x^*(t) - \tfrac{1}{2}u^*(t)^2\big)\,dt = \int_0^T \big(Tt - \tfrac{1}{2}t^2 + x_0 - \tfrac{1}{2}(T - t)^2\big)\,dt \qquad (*)$$

Integrating we find $V(x_0, T) = \big|_{t=0}^{t=T}(\frac{1}{2}Tt^2 - \frac{1}{6}t^3 + x_0t + \frac{1}{6}(T - t)^3) = \frac{1}{6}T^3 + x_0T$, so $\partial V/\partial x_0 = T = p(0)$. (If we were asked only to find $\partial V/\partial x_0$, it would be easier to differentiate $(*)$ with respect to x_0 under the integral sign: $\partial V/\partial x_0 = \int_0^T dt = T$.)

The value of the Hamiltonian "along the optimal path" is $H^*(t) = H(t, x^*(t), u^*(t), p(t)) = x^*(t) - \frac{1}{2}u^*(t)^2 + p(t)u^*(t) = Tt - \frac{1}{2}t^2 + x_0 - \frac{1}{2}(T - t)^2 + (T - t)^2$, and so $H^*(T) = \frac{1}{2}T^2 + x_0$. We see that $\partial V/\partial T = \frac{1}{2}T^2 + x_0 = H^*(T)$.

9.6.4 (a) The Hamiltonian is $H(t, x, u, p) = 2x^2e^{-2t} - ue^t + pue^t = 2x^2e^{-2t} + (p - 1)ue^t$. (Recall that $x(T)$ is free.) Suppose that $(x^*(t), u^*(t))$ is an admissible pair with adjoint function $p(t)$. Since $H'_x = 4xe^{-2t}$, the differential equation for $p(t)$ is $\dot{p}(t) = -4e^{-2t}x^*(t)$. For all t in $[0, T]$, we have $\dot{x}^*(t) = u^*(t)e^t \geq 0$, and hence $x^*(t) \geq x^*(0) = 1$ for all t in $[0, T]$. Therefore, $\dot{p}(t) < 0$, so $p(t)$ is strictly decreasing. From the maximum principle, for every t in $[0, T]$, $u^*(t)$ is the value of u in $[0, 1]$ that maximizes $H(t, x^*(t), u, p(t)) = 2(x^*(t))^2e^{-2t} + (p(t) - 1)ue^t$. Since $2(x^*(t))^2e^{-2t}$ does not depend on u, $u^*(t)$ is the value of u in $[0, 1]$ that maximizes $(p(t) - 1)u$. Thus, $u^*(t) = 1$ if $p(t) > 1$, $u^*(t) = 0$ if $p(t) < 1$. Since $p(T) = 0$, we have $p(t) < 1$ for t close to T. There are two possibilities:

Case A: Suppose $p(0) \leq 1$. Since p is strictly decreasing, we get $p(t) < 1$, and hence $u^*(t) = 0$ for all $t > 0$. It follows that $\dot{x}^*(t) = u^*(t) = 0$ and $x^*(t) \equiv x^*(0) = 1$ for all t in $[0, T]$.

It remains to determine $p(t)$. We have $\dot{p}(t) = -4e^{-2t}x^*(t) = -4e^{-2t}$, and so $p(t) = 2e^{-2t} + C$,

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where the constant C is determined by $p(T) = 0$, and hence $C = -2e^{-2T}$. It follows that $p(t) = 2(e^{-2t} - e^{-2T}) = 2e^{-2t}(1 - e^{-2(T-t)})$. Since we have assumed $p(0) \leq 1$, we must have $2(1 - e^{-2T}) \leq 1$. This gives $e^{-2T} \geq 1/2$, and so $-2T \geq \ln(1/2) = -\ln 2$, or $T \leq \frac{1}{2}\ln 2$.

Case B: Suppose $p(0) > 1$. Then there exists a point $t^* \in (0, T)$ with $p(t^*) = 1$. In this case $p(t) > 1$ for $t < t^*$ and $p(t) < 1$ for $t > t^*$. This gives

$$u^*(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq t^* \\ 0 & \text{if } t^* < t \leq 1 \end{cases}$$

Since $\dot{x}^*(t) = u^*(t)$ and $x^*(0) = 1$, we get $x^*(t) = e^t$ if $0 \leq t \leq t^*$ and $x^*(t) = e^{t^*}$ if $t^* \leq t \leq 1$. Thus, $\dot{p}(t) = -4e^{-2t}x^*(t) = -4e^{-t}$ if $0 \leq t < t^*$, and $\dot{p}(t) = -4e^{t^*-2t}$ if $t^* < t \leq 1$. Integration gives

$$p(t) = \begin{cases} 4e^{-t} + C_1 & \text{if } 0 \leq t < t^* \\ 2e^{t^*-2t} + C_2 & \text{if } t^* < t \leq 1 \end{cases}$$

Since p is continuous, both expressions for $p(t)$ are valid for $t = t^*$. Moreover, $p(T) = 0$, so we get the equations

$$\text{(i) } p(t^*) = 1 = 4e^{-t^*} + C_1 \quad \text{(ii) } p(t^*) = 1 = 2e^{-t^*} + C_2 \quad \text{(iii) } p(T) = 0 = 2e^{t^*-2T} + C_2$$

From (i) and (iii) we get $C_1 = 1 - 4e^{-t^*}$ and $C_2 = -2e^{t^*-2T}$. Hence,

$$p(t) = \begin{cases} 4(e^{-t} - 4e^{-t^*}) + 1 & \text{if } 0 \leq t \leq t^* \\ 2e^{t^*}(e^{-2t} - e^{-2T}) & \text{if } t^* \leq t \leq 1 \end{cases}$$

It remains to determine t^*. From (ii) and (iii) we get $1 = 2e^{-t^*} - 2e^{t^*-2T}$. Multiplying this equation by e^{t^*} yields

$$e^{t^*} = 2 - 2e^{2t^*-2T} \tag{iv}$$

Multiplying with $\frac{1}{2}e^{2T}$ and rearranging gives $(e^{t^*})^2 + \frac{1}{2}e^{t^*}e^{2T} - e^{2T} = 0$. This is a second degree equation for determining e^{t^*}, and we find

$$e^{t^*} = -\tfrac{1}{4}e^{2T} + \sqrt{\tfrac{1}{16}e^{4T} + e^{2T}} \quad \text{and so} \quad t^* = \ln\left(\sqrt{\tfrac{1}{16}e^{4T} + e^{2T}} - \tfrac{1}{4}e^{2T}\right)$$

(Since e^{t^*} is positive, we must take the positive square root.) This solution makes sense if and only if $t^* > 0$, i.e. if and only if $\sqrt{\frac{1}{16}e^{4T} + e^{2T}} - \frac{1}{4}e^{2T} > 1$. This inequality is equivalent to $\frac{1}{16}e^{4T} + e^{2T} > (\frac{1}{4}e^{2T} + 1)^2 = \frac{1}{16}e^{4T} + \frac{1}{2}e^{2T} + 1$, and so $e^{2T} > 2 \iff T > \frac{1}{2}\ln 2$. For the summing up see the book.

(b) We have to consider two cases A and B. Note that in both cases $u^*(T) = 0$ (and $p(T) = 0$), and so $H^*(T) = H(T, x^*(T), u^*(T), p(T)) = 2x^*(T)^2e^{-2T}$.

Case A: If $T \leq \frac{1}{2}\ln 2$, then $x^*(T) = 1$, $H^*(T) = 2e^{-2t}$, and

$$V(T) = \int_0^T \big(2(x^*(t))^2e^{-2t} - u^*(t)e^t\big)\,dt = \int_0^T 2e^{-2t}\,dt = \Big|_0^T -e^{-2t} = 1 - e^{-2T}$$

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This implies $V'(T) = 2e^{-2T} = H^*(T)$.

Case B: For $T > \frac{1}{2}\ln 2$, we have $x^*(T) = x^*(t^*) = e^{t^*}$. This gives $H^*(T) = 2e^{2t^*-2T}$. Furthermore,

$$V(T) = \int_0^T \big(2(x^*(t))^2 e^{-2t} - u^*(t)e^t\big)\,dt$$
$$= \int_0^{t^*} (2 - e^t)\,dt + \int_{t^*}^T 2e^{2t^*-2t}\,dt = 2t^* - e^{t^*} + 1 - e^{2t^*-2T} + 1$$

Note that t^* depends of T. Hence we get

$$V'(T) = 2\frac{dt^*}{dT} - e^{t^*}\frac{dt^*}{dT} - e^{2t^*-2T}(2\frac{dt^*}{dT} - 2) = (2 - e^{t^*} - 2e^{2t^*-2T})\frac{dt^*}{dT} + 2e^{2t^*-2T}$$

Equation (iv) shows that $2 - e^{t^*} - 2e^{2t^*-2T} = 0$, and therefore

$$V'(T) = 0 + 2e^{2t^*-2T} = H^*(T)$$

as expected. (Alternatively we could have inserted the expressions found for t^* and e^{t^*} into the expression for $V(T)$, and found $V(T)$ in terms of T. The algebra required is heavy.)

9.7

9.7.1 (a) The Hamiltonian $H = 100 - x - \frac{1}{2}u^2 + pu$ is concave in (x, u) and $u \in (-\infty, \infty)$, so the following conditions are sufficient for optimality:

(i) $(H'_u)^* = -u^*(t) + p(t) = 0$;
(ii) $\dot{p}(t) = 1$;
(iii) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = x_0$, and $x^*(1) = x_1$.

From (ii) we get $p(t) = t + A$, and (i) and (iii) give $\dot{x}^*(t) = u^*(t) = p(t) = t + A$. Integrating and using the two boundary conditions gives $B = x_0$, $A = x_1 - x_0 - \frac{1}{2}$, so the solution is as given in the book.

(b) $V = V(x_0, x_1) = \displaystyle\int_0^1 (100 - x^*(t) - \tfrac{1}{2}u^*(t)^2)\,dt$

$$= \int_0^1 \Big[100 - \tfrac{1}{2}t^2 - (x_1 - x_0 - \tfrac{1}{2})t - x_0 - \tfrac{1}{2}\big(t + x_1 - x_0 - \frac{1}{2}\big)^2\Big]\,dt.$$

Differentiating w.r.t. x_0 under the integral sign yields

$\partial V/\partial x_0 = \displaystyle\int_0^1 (t - 1 + t + x_1 - x_0 - \tfrac{1}{2})\,dt = \int_0^1 (2t - \tfrac{3}{2} + x_1 - x_0)\,dt = \Big|_0^1 (t^2 - \tfrac{3}{2}t + (x_1 - x_0)t) =$ $x_1 - x_0 - \frac{1}{2} = p(0)$, as it should be. In the same way, $\partial V/\partial x_1 = \displaystyle\int_0^1 [-t - \big(t + x_1 - x_0 - \tfrac{1}{2}\big)]\,dt =$ $\displaystyle\int_0^1 (-2t - x_1 + x_0 + \tfrac{1}{2})\,dt = \Big|_0^1 (-t^2 - x_1 t + x_0 t + \tfrac{1}{2}t) = -x_1 + x_0 - \tfrac{1}{2} = -p(1)$, as it should be.

9.7.2 (a) The Hamiltonian is $H = (1 - s)\sqrt{k} + ps\sqrt{k} = \sqrt{k} + \sqrt{k}(p - 1)s$. (In (b) we use the Arrow theorem, so we assume $p_0 = 1$.) The maximum principle (Theorem 9.4.1) gives the conditions:

(i) $s = s^*(t)$ maximizes $\sqrt{k^*(t)} + \sqrt{k^*(t)}(p(t) - 1)s$ for $s \in [0, 1]$;
(ii) $\dot{p}(t) = -H'_k(t, k^*(t), s^*(t), p(t)) = -\dfrac{1}{2\sqrt{k^*(t)}}[1 + s^*(t)(p(t) - 1)]$, $p(10) = 0$;

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(iii) $\dot{k}^*(t) = s^*(t)\sqrt{k^*(t)}, \quad k^*(0) = 1.$

Since $s^*(t) \geq 0$, it follows from (iii) that $k^*(t) \geq 1$ for all t. Then we see from (i) that

$$s^*(t) = \begin{cases} 1 & \text{if } p(t) > 1 \\ 0 & \text{if } p(t) < 1 \end{cases} \tag{iv}$$

By studying (ii) and (iv) we see that $\dot{p}(t) < 0$ for all t. (If $p(t) > 1$, then $s^*(t) = 1$ and from (ii), $\dot{p}(t) = -p(t)/2\sqrt{k^*(t)} < 0$. (What if $p(t) < 1$?) Thus $p(t)$ is strictly decreasing.

Suppose $p(t) < 1$ for all t in [0, 10]. Then $s^*(t) \equiv 0$ and $k^*(t) \equiv 1$. From (ii) this gives $\dot{p}(t) = -1/2$, and with $p(10) = 0$ we get $p(t) = 5-t/2$. But then $p(t) > 1$ for $t < 8$, a contradiction. We conclude that there must exist a t^* in (0, 10) such that $p(t^*) = 1$. Then $s^*(t) = 1$ on $[0, t^*]$ and $s^*(t) = 0$ on $(t^*, 10]$. But then $\dot{k}^*(t) = \sqrt{k^*(t)}$ on $[0, t^*]$ and $\dot{k}^*(t) = 0$ on $[0, t^*]$. By integrating the differential equation for $k^*(t)$ on $[0, t^*]$, we find $2\sqrt{k^*(t)} = t + C$. With $k^*(0) = 1$ we get $C = 2$, so $k^*(t) = (\frac{1}{2}t + 1)^2$. On $(t^*, 10]$ we have $k^*(t) = (\frac{1}{2}t^* + 1)^2$ since $k^*(t)$ is continuous. Thus,

$$s^*(t) = \begin{cases} 1 & \text{if } [0, t^*] \\ 0 & \text{if } (t^*, 10] \end{cases}, \qquad k^*(t) = \begin{cases} (\frac{1}{2}t + 1)^2 & \text{if } [0, t^*] \\ (\frac{1}{2}t^* + 1)^2 & \text{if } (t^*, 10] \end{cases} \tag{v}$$

On $(t^*, 10]$ we get from (ii) that $\dot{p}(t) = -1/2\sqrt{k^*(t)} = -1/(t^*+2)$, and so $p(t) = -t/(t^*+2)+D$. Since $p(10) = 0$, this implies that $p(t) = (10-t)/(t^*+2)$ on $(t^*, 10]$. But $p(t^*) = 1$, so $(10-t)/(t^*+2) = 1$, from which it follows that $t^* = 4$. It remains only to find $p(t)$ on $[0, t^*]$. On this interval $\dot{p}(t) = -p(t)/2\sqrt{k^*(t)} = -p(t)/(t+2)$. The solution of this separable equation is $p(t) = E/(t+2)$, for some constant E. But $p(4) = 1$, so $E = 6$. We have found the only possible solution. See the answer in the book.

(b) See the answer in the book.

9.7.3 (a) The Hamiltonian $H(t, x, u, p) = e^{-\beta t}\sqrt{u} + p(\alpha x(t) - u)$ is concave in (x, u), so according to Theorem 9.7.1, the following conditions are sufficient for optimality:

(i) $u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t)) = e^{-\beta t}\sqrt{u} + p(t)(\alpha x^*(t) - u)$ for $u \geq 0$;
(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -\alpha p(t)$;
(iii) $\dot{x}^*(t) = \alpha x^*(t) - u^*(t), \; x^*(0) = 1, x^*(T) = 0.$

From (ii) we get $p(t) = Ae^{-\alpha t}$. Define the function g by $g(u) = e^{-\beta t}\sqrt{u} - Ae^{-\alpha t}u$. Then $g'(u) = e^{-\beta t}(1/2\sqrt{u}) - Ae^{-\alpha t}$, and we see that $g'(u) > 0$ when u is slightly larger than 0. This means that $u^*(t) = 0$ cannot maximize g, or the Hamiltonian, for any t. Hence $u^*(t) > 0$ and $g'(u^*(t)) = 0$, so

$$\frac{e^{-\beta t}}{2\sqrt{u^*(t)}} = Ae^{-\alpha t} \quad \text{or} \quad u^*(t) = \frac{1}{4A^2}e^{2(\alpha-\beta)t}$$

From (iii) we get the following linear differential equation for $x^*(t)$:

$$\dot{x}^*(t) = \alpha x^*(t) - \frac{1}{4A^2}e^{2(\alpha-\beta)t} \quad \text{with solution} \quad x^*(t) = Ce^{\alpha t} - \frac{1}{4A^2(\alpha - 2\beta)}e^{2(\alpha-\beta)t}$$

The two constants A and C are determined by the boundary conditions in (iii). The explicit expressions for $x^*(t)$, $u^*(t)$, and $p(t)$ can be found in the answer in the book. (There is a misprint in the formula for $x^*(t)$: the first fraction must be multiplied by $e^{\alpha t}$.)

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(b) If $x(T) = 0$ is replaced by $x(T) \geq 0$, the only change in the conditions (i)–(iii) is that (ii) is augmented by $p(T) \geq 0$, with $p(T) = 0$ if $x^*(T) > 0$. Again $p(t) = Ae^{-\alpha t}$, so $p(T) = 0$ would imply $Ae^{-\alpha T} = 0$, so $A = 0$ and $p(t) \equiv 0$. The maximum condition (i) would then imply that $u^*(t)$ for $u \geq 0$ would maximize $e^{-\beta t}\sqrt{u}$, which has no maximum. Hence $x^*(T) = 0$, and the solution is as in (a).

9.8

9.8.1 (b) In this problem we use Theorem 9.8.1 and we try to find the only possible solution to the problem. The Hamiltonian $H = p_0(-9 - \frac{1}{4}u^2) + p(t)u$ is concave in (x, u), but this does not guarantee that a solution to the necessary conditions is optimal. (See Note 9.8.1.)

Suppose $(x^*(t), u^*(t))$ is an optimal pair defined on $[0, t^*]$. Then there exists a continuous function $p(t)$ and a number p_0, which is either 0 or 1, such that for all t in $[0, t^*]$ we have $(p_0, p(t)) \neq (0, 0)$ and

(i) $u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t)) = p_0(-9 - \frac{1}{4}u^2) + p(t)u$ for $u \in \mathbb{R}$;

(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = 0$;

(iii) $\dot{x}^*(t) = u^*(t),\ x^*(0) = 0, x^*(T) = 16$;

(iv) $H(t^*, x^*(t^*), u^*(t^*), p(t^*)) = p_0(-9 - \frac{1}{4}u^*(t^*)^2) + p(t)u^*(t^*) = 0$.

Since H is concave in (x, u), condition (i) is equivalent to $(H'_u)^* = -p_0\frac{1}{2}u^*(t) + p(t) = 0$. Then $p_0 = 0$ implies $p(t) = 0$, which contradicts $(p_0, p(t)) \neq (0, 0)$. Hence $p_0 = 1$, and $u^*(t) = 2p(t)$. From (ii) we have $p(t) = \bar{p}$ for some constant $\bar{p}$, and so $u^*(t) = 2\bar{p}$. Moreover, $\dot{x}^*(t) = u^*(t) = 2\bar{p}$. Integrating and using $x^*(0) = 0$ this gives $x^*(t) = 2\bar{p}t$, and $x^*(t^*) = 16$ yields $\bar{p} = 8/t^*$. Finally, (iv) implies $-9 - \frac{1}{4}(2\bar{p})^2 + \bar{p}2\bar{p} = 0$, or $\bar{p}^2 = 9$. Here $\bar{p} = -3$ gives $t^* = 8/\bar{p} < 0$. So the only possible solution is $\bar{p} = 3$, Then $t^* = 8/3$, $u^*(t) = 6$, and $x^*(t) = 6t$.

9.8.2 Consider the case $B \geq aT^2/4b$. Then from Problem 9.4.7, $u^*(t) = a(2t - T)/4b + B/T$ and $x^*(t) = at(t - T)/4b + Bt/T$ and $p(t) = 2bu^*(t)$. To determine the optimal T^*, the condition is $H(T^*) = 0$, or $ax^*(T^*) + b(u^*(T^*))^2 = p(T^*)u^*(T^*) = 2b(u^*(T^*))^2$. This reduces to $aB/b = (u^*(T^*))^2$, or $u^*(T^*) = \sqrt{aB/b}$, that is $aT^*/4b + B/T^* = \sqrt{aB/b}$. Solving this equation for T^* gives the unique solution $T^* = 2\sqrt{bB/a}$. (Note that this is the positive solution T of $B = aT^2/4b$.)

If $B < aT^2/4b$, we find that the equation $H^*(T^*) = 0$ does not determine T^*. (To save on storage costs, the firm waits until $t = T - 2\sqrt{Bb/a}$ to start production, and then produces at an optimal rate until delivery time T. Note that no discounting is assumed, so waiting costs nothing.)

9.9

9.9.2 The current value Hamiltonian $H^c = 10u - u^2 - 2 - \lambda u$ is concave in (x, u), so the following conditions are sufficient for optimality:

(i) $u = u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t)) = 10u - u^2 - 2 - \lambda(t)u$ for $u \geq 0$;

(ii) $\dot{\lambda}(t) - 0.1\lambda = -(H^c)'_x(t, x^*(t), u^*(t), \lambda(t)) = 0$;

(iii) $\lambda(5) \geq 0$, with $\lambda(5) = 0$ if $x^*(5) > 0$;

(iv) $\dot{x}^*(t) = -u^*(t),\ x^*(0) = 10, x^*(5) \geq 0$.

From (ii) we have $\lambda(t) = Ae^{0.1t}$ for some constant A. Suppose $u^*(t) > 0$ for all t in $[0, 5]$. Then (i) is equivalent to $(H^c)'_u(t, x^*(t), u^*(t), \lambda(t)) = 10 - 2u^*(t) - Ae^{0.1t} = 0$, that is,

(v) $u^*(t) = 5 - \frac{1}{2}Ae^{0.1t}$.

Then $\dot{x}^*(t) = -5 + \frac{1}{2}Ae^{0.1t}$, and integration gives $x^*(t) = -5t + 5Ae^{0.1t} + B$, where B is a new constant. The initial condition $x^*(0) = 10$ yields $B = 10 - 5A$, so $x^*(t) = 5A(e^{0.1t} - 1) - 5t + 10$.

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Next look at condition (iii) which reduces to $A \geq 0$, with $A = 0$ if $x^*(5) > 0$. But $A = 0$ would imply $x^*(t) = -5t + 10$, which for $t = 5$ yields the contradiction $x^*(5) < 0$. Thus $A > 0$ and $x^*(5) = 0$. But then $0 = 5A(e^{0.5} - 1) - 15$, so $A = 3/(e^{0.5} - 1)$, and the same solutions for $x^*(t)$ and $u^*(t)$ as in Problem 9.4.6 are found. Since all the conditions (i)–(iv) are satisfied by the admissible pair $(x^*(t), u^*(t))$, with the given $\lambda(t)$, we have found the solution.

9.9.3 The current value Hamiltonian $H^c = -2u - u^2 + \lambda u$ is concave in (x, u), and $u \in \mathbb{R}$, so the following conditions are sufficient for optimality:

(i) $(H^c)'_u = -2 - 2u^*(t) + \lambda = 0$;
(ii) $\dot{\lambda}(t) - 0.1\lambda = -(H^c)'_x(t, x^*(t), u^*(t), \lambda(t)) = 0$;
(iii) $\dot{x}^*(t) = u^*(t),\ x^*(0) = 1,\ x^*(1) = 0$.

From (ii) we have $\lambda(t) = Ae^{0.1t}$ for some constant A. From (ii) we have $u^*(t) = \frac{1}{2}\lambda(t) - 1 = \frac{1}{2}Ae^{0.1t} - 1$. Then $\dot{x}^*(t) = \frac{1}{2}Ae^{0.1t} - 1$, with solution $x^*(t) = 5Ae^{0.1t} - t + B$. The constants A and B are determined from the boundary conditions, and we get the same solution as in Problem 9.5.3.

9.10

9.10.2 With A as the state variable, the Hamiltonian is $H(t, A, u, p) = U(rA(t) + w - u(t))e^{-\rho t} + pu$, and the scrap value function is $S(T, A) = e^{-\rho T}\varphi(A)$. (We assume that $\lambda_0 = 1$.) Moreover, w is a constant. With the assumptions in Example 8.5.3, the utility function U has $U' > 0$ and $U'' < 0$. This implies that $H(t, A, u, p)$ is concave in (A, u). Since $\varphi(A)$ is also concave, a set of sufficient conditions for optimality is then (assuming interior solution in (i)):

(i) $H'_3(t, A^*(t), u^*(t), p(t)) = 0$, or $p(t) = U'(rA^*(t) + w - u^*(t))e^{-\rho t}$;
(ii) $\dot{p} = -H'_2(t, A^*(t), u^*(t), p(t)) = -rU'(rA^*(t) + w - u^*(t))e^{-\rho t}$;
(iii) $p(T) = S'_A(T, A^*(T)) = e^{-\rho T}\varphi'(A^*(T))$;
(iv) $\dot{A}^*(t) = u^*(t),\ A^*(0) = A_0$.

In the answer in the book we go a little further. We differentiate the expression for $p(t)$ in (i) w.r.t. t and equate it to the expression for $\dot{p}(t)$ in (ii). Using $u^*(t) = \dot{A}^*(t)$, this gives

$$U''(rA^* + w - \dot{A}^*)(r\dot{A}^* - \ddot{A}^*)e^{-\rho t} - \rho U'(rA^* + w - \dot{A}^*)e^{-\rho t} = -rU'(rA^* + w - \dot{A}^*)e^{-\rho t}$$

Multiplying by $-e^{\rho t}$ and rearranging gives $\ddot{A}^* - r\dot{A}^* + (\rho - r)U'/U'' = 0$. Combining (i) and (iii), we also see that $\varphi'(A^*(T)) = U'(rA^*(T) + w - u^*(T))$.

9.10.3 Compared with Problem 9.7.2 the only difference is that instead of the condition "$x(10)$ free", we have now included a scrap value in the objective function. The scrap value function is $S(k) = 10\sqrt{k}$, with $S'(k) = 5/\sqrt{k}$. Conditions (i)–(iv) in the answer to Problem 9.7.2 are still valid except that $p(10) = 0$ in (ii) is now replaced by

$$p(10) = \frac{5}{\sqrt{k^*(10)}} \qquad \text{(ii)}'$$

Again $p(t)$ is strictly decreasing, and $p(t) < 1$ for all t in $[0, 10]$ is again easily seen to be impossible. Suppose $p(t) > 1$ for all t in $[0, 10]$. (This was not an option when we required $p(10) = 0$.) Then $s^*(t) \equiv 1$ and $\dot{k}^*(t) = \sqrt{k^*(t)}$, with $k^*(0) = 1$. It follows that $2\sqrt{k^*(t)} = t + 2$, or $k^*(t) = (\frac{1}{2}t + 1)^2$. In particular, $k^*(10) = 36$. Then (ii)′ gives $p(10) = 5/6 < 1$ a contradiction. We conclude that there must exist a t^* in $(0, 10)$ such that $p(t^*) = 1$, and (v) in the answer to Problem 9.7.2 is still valid (although

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with a different t^*).

On $(t^*, 10]$ we again get $p(t) = -t/(t^*+2) + D$ and $1 = p(t^*) = -t^*/(t^*+2)$, which implies that $D = (2t^*+2)/(t^*+2)$, and thus $p(t) = -t/(t^*+2) + (2t^*+2)/(t^*+2)$. In particular, $p(10) = (2t^*-8)/(t^*+2)$. We now use (ii)$'$ to determine t^*:

$$\frac{2t^*-8}{t^*+2} = \frac{5}{\frac{1}{2}t^*+1} = \frac{10}{t^*+2}$$

from which it follows that $t^* = 9$. The rest is routine, see the answer in the book. (Again the Arrow condition is satisfied, and it is valid also in this case.)

9.10.4 (a) This is a problem with scrap value function $S(x) = \frac{1}{2}x$. We use Theorem 9.10.1. Let $(x^*(t), u^*(t))$ be an admissible pair. With $p_0 = 1$, the Hamiltonian is $H(t, x, u, p) = x - u + pu = x + (p-1)u$. (Note that the scrap value function is not to be included in the Hamiltonian.) If $(x^*(t), u^*(t))$ is an optimal pair in the problem, and $p(t)$ is the adjoint function, then according to (B) in Theorem 9.10.1,

$$\dot{p}(t) = -\frac{\partial H^*}{\partial x} = -1 \quad \text{and} \quad p(1) = S'(x^*(1)) = \frac{1}{2} \tag{i}$$

Moreover, $u^*(t) = 1$ if $p(t) > 1$ and $u^*(t) = 0$ if $p(t) < 1$. From (i) we get

$$p(t) = \tfrac{3}{2} - t$$

We see that the strictly decreasing function $p(t)$ is equal to 1 at $t = 1/2$. Hence

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2] \\ 0 & \text{if } t \in (1/2, 1] \end{cases}$$

Since $\dot{x}^*(t) = u^*(t)$ and $x(0) = 1/2$, we get

$$x^*(t) = \begin{cases} t + 1/2 & \text{if } t \in [0, 1/2] \\ 1 & \text{if } t \in (1/2, 1] \end{cases}$$

We have found the optimal solution because $H(t, x, u, p(t))$ and $S(x) = x/2$ are concave (in fact linear) functions of (x, u).

(b) The scrap value function is now $\bar{S}(x) = -\frac{1}{4}(x-2)^2$, but the Hamiltonian is as in (a). Condition (i) is replaced by

$$\dot{p}(t) = -1 \quad \text{and} \quad p(1) = \bar{S}'(x^*(1)) = -\tfrac{1}{2}(x^*(1) - 2) = 1 - \tfrac{1}{2}x^*(1) \tag{ii}$$

From (ii) we see that $p(t) = -t + C$ and since $p(1) = 1 - \frac{1}{2}x^*(1)$, we have

$$p(t) = -t + 2 - \tfrac{1}{2}x^*(1) \tag{iii}$$

Since $x^*(0) = 1/2$ and $0 \le \dot{x}^* \le 1$, it is clear that $1/2 \le x^*(1) \le 3/2$. Condition (iii) therefore gives

$$1/4 \le p(1) \le 3/4$$

Since $\dot{p}(t) = -1$ for all t, we have $p(t) = p(0) - t$. Hence, $p(0) = p(1) + 1$ and

$$5/4 \le p(0) \le 7/4$$

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Condition (ii) is still valid, so there exists a unique number t^* between 0 and 1 such that $p(t^*) = 1$, and

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, t^*], \\ 0 & \text{if } t \in (t^*, 1], \end{cases} \qquad x^*(t) = \begin{cases} 1/2 + t & \text{if } t \in [0, t^*] \\ 1/2 + t^* & \text{if } t \in (t^*, 1] \end{cases}$$

To determine t^* we use the transversality condition. We know that $p(t) = p(0) - t = p(1) + 1 - t$, so $p(1) = p(t) + t - 1$ for all t. Since $p(t^*) = 1$, we get $p(1) = t^*$. The transversality condition then gives $t^* = p(1) = 1 - \frac{1}{2}x^*(1) = 1 - \frac{1}{2}\left(\frac{1}{2} + t^*\right) = \frac{3}{4} - \frac{1}{2}t^*$, which means that $t^* = \frac{1}{2}$. The optimal pair $(x^*(t), u^*(t))$ is therefore exactly as in (a), but this is accidental. A different scrap value function would usually have given another solution.

9.10.5 (a) The Hamiltonian $H(t, x, u, p) = -u^2 - px + pu$ is concave in (x, u) and the scrap value function $S(x) = -x^2$ is concave in x. The following conditions are therefore sufficient for optimality:

(i) $H_3'(t, x^*(t), u^*(t), p) = -2u^*(t) + p(t) = 0, \quad \text{so} \quad u^*(t) = \frac{1}{2}p(t)$;

(ii) $\dot{p}(t) = -\partial H^*/\partial x = p(t) \quad \text{and} \quad p(T) = -S'(x^*(T)) = -2x^*(T)$;

(iii) $\dot{x}^*(t) = -x^*(t) + u^*(t), x^*(0) = x_0$.

From (ii) we get $p(t) = Ce^t$ for an appropriate constant C, and since $p(T) = -2x^*(T)$, we get $p(t) = -2x^*(T)e^{t-T}$ and also $u^*(t) = \frac{1}{2}p(t) = -x^*(T)e^{t-T}$. The differential equation

$$\dot{x}^*(t) = -x^*(t) + u^*(t) = -x^*(t) - x^*(T)e^{t-T}$$

has the general solution

$$x^*(t) = De^{-t} - \tfrac{1}{2}x^*(T)e^{t-T}$$

From $x^*(T) = De^{-T} - \frac{1}{2}x^*(T)$ we get $D = \frac{3}{2}x^*(T)e^T$. The initial condition $x^*(0) = x_0$ gives

$$D - \tfrac{1}{2}x^*(T)e^{-T} = \tfrac{3}{2}x^*(T)e^T - \tfrac{1}{2}x^*(T)e^{-T} = \tfrac{1}{2}x^*(T)(3e^T - e^{-T}) = x_0$$

so

$$x^*(T) = \frac{2x_0}{3e^T - e^{-T}} = \frac{2x_0e^T}{3e^{2T} - 1} \quad \text{and} \quad D = \frac{3}{2}x^*(T)e^T = \frac{3x_0e^{2T}}{3e^{2T} - 1}$$

Thus the optimal solution is

$$u^*(t) = -\frac{2x_0e^t}{3e^{2T} - 1}, \quad x^*(t) = \frac{x_0(3e^{2T-t} - e^t)}{3e^{2T} - 1}, \quad p(t) = -\frac{4x_0e^t}{3e^{2T} - 1}$$

(b) We have

$$\begin{aligned} V(x_0, T) &= -\int_0^T (u^*(t))^2\,dt - (x^*(T))^2 = -\frac{4x_0^2}{(3e^{2T} - 1)^2}\left[\int_0^T e^{2t}\,dt + e^{2T}\right] \\ &= -\frac{4x_0^2}{(3e^{2T} - 1)^2}\left[\Big|_0^T \frac{1}{2}e^{2t} + e^{2T}\right] = -\frac{4x_0^2}{(3e^{2T} - 1)^2}\left[\frac{3}{2}e^{2T} - \frac{1}{2}\right] = -\frac{2x_0^2}{3e^{2T} - 1} \end{aligned}$$

It follows that

$$\frac{\partial V}{\partial x_0} = -\frac{4x_0}{3e^{2T} - 1} = p(0) \quad \text{and} \quad \frac{\partial V}{\partial T} = \frac{12x_0^2e^{2T}}{(3e^{2T} - 1)^2}$$

We see from the solutions in (a) that $p(T) = -2x^*(T)$ and $u^*(t) = -x^*(T)$, so $H^*(T) = -u^*(T)^2 + p(T)(-x^*(T) + u^*(T)) = 3x^*(T)^2 = \partial V/\partial T$.

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9.10.6 The current value Hamiltonian $H^c(t, x, u, \lambda) = -(x-u)^2 + \lambda(u - x + a)$ is concave in (x, u). The scrap value function $S(x) = -x^2$ is concave in x. The following conditions are therefore sufficient for an admissible pair (x^*, u^*) to solve the problem:

(i) $\partial(H^c)^*/\partial u = 0$, i.e. $\lambda(t) = -2(x^*(t) - u^*(t))$;
(ii) $\dot{\lambda}(t) - r\lambda = -\partial(H^c)^*/\partial x = 2(x^*(t) - u^*(t)) + \lambda(t)$, and $\lambda(T) = -2x^*(T)$;
(iii) $\dot{x}^*(t) = u^*(t) - x^*(t) + a$, $x^*(0) = 0$.

From (i) and (ii) it follows that $\dot{\lambda}(t) = r\lambda$, so $\lambda(t) = Ae^{rt}$, where A is a constant. Then (i) yields $x^*(t) - u^*(t) = -\frac{1}{2}Ae^{rt}$. But then $\dot{x}^*(t) = u^*(t) - x^*(t) + a = \frac{1}{2}Ae^{rt} + a$, so $x^*(t) = (A/2r)e^{rt} + at + B$. The initial condition $x^*(0) = 0$ gives $B = -A/2r$, so

$$x^*(t) = \frac{A}{2r}(e^{rt} - 1) + at \tag{iv}$$

From (ii), $x^*(T) = -\frac{1}{2}\lambda(T) = -\frac{1}{2}Ae^{rT}$, and so (iv) with $t = T$ yields $-\frac{1}{2}Ae^{rT} = (A/2r)(e^{rT} - 1) + aT$. Solving for A yields

$$A = -\frac{2arT}{e^{rT}(1+r) - 1}$$

The expressions for $\lambda(t)$, $x^*(t)$, and $u^*(t)$ follow easily. See the answer in the book.

9.11

9.11.1 The current value Hamiltonian $H^c = \ln u - \lambda(0.1x - u)$ is concave in (x, u), and $u > 0$, so the following conditions are sufficient for optimality:

(i) $(H^c)'_u(t, x^*, u^*, \lambda(t)) = 1/u^*(t) + \lambda = 0$;
(ii) $\dot{\lambda}(t) - 0.2\lambda = -(H^c)'_x(t, x^*, u^*, \lambda(t)) = -0.1\lambda$;
(iii) (a) $\lim_{t\to\infty} \lambda(t)e^{-0.2t}(-x^*(t)) \geq 0$;
(b) There exists a number M such that $|\lambda(t)e^{-0.2t}| \leq M$ for all $t \geq 0$;
(c) There exists a number t' such that $\lambda(t) \geq 0$ for all $t \geq t'$;
(iv) $\dot{x}^*(t) = 0.1x^*(t) - u^*(t)$, $x^*(0) = 10$, $\lim_{t\to\infty} x^*(t) \geq 0$.

From (ii) we get $\dot{\lambda}(t) = 0.1\lambda(t)$, and thus $\lambda(t) = Ae^{0.1t}$, for some constant A. Condition (i) yields $u^*(t) = 1/\lambda(t) = e^{-0.1t}/A$. Then from (iv), $\dot{x}^*(t) = 0.1x^*(t) - e^{-0.1t}/A$. The solution of this linear differential equation, with $x^*(0) = 10$, is easily seen to be $x^*(t) = (10 - 5/A)e^{0.1t} + 5e^{-0.1t}/A$. Condition (iv) requires $\lim_{t\to\infty} x^*(t) = \lim_{t\to\infty}[(10 - 5/A)e^{0.1t} + 5e^{-0.1t}/A] \geq 0$. This obviously requires $10 - 5/A \geq 0$. From (iii)(c) we see that we must have $A \geq 0$, and so $A \geq 1/2$. However, referring to (iii)(a),

$$\lambda(t)e^{-0.2t}(-x^*(t)) = Ae^{0.1t}e^{-0.2t}(-(10 - 5/A)e^{0.1t} - 5e^{-0.1t}/A) = -(10A - 5) - 5e^{-0.2t}$$

The limit of this expression can be ≥ 0 only if $10A - 5 \leq 0$, i.e. $A \leq 1/2$. Thus we must have $A = 1/2$. Then $u^*(t) = 2e^{-0.1t}$, $x^*(t) = 10e^{-0.1t}$, with $\lambda(t) = \frac{1}{2}e^{0.1t}$.

It remains to check that the conditions in (iii) are satisfied. First, (iii)(a) is satisfied because we have $\lim_{t\to\infty} \frac{1}{2}e^{0.1t}e^{-0.2t}(-10e^{-0.1t}) = -5\lim_{t\to\infty} e^{-0.2t} = 0$. Since $|\lambda(t)e^{-0.2t}| = |\frac{1}{2}e^{0.1t}e^{-0.2t}| = \frac{1}{2}e^{-0.1t} \leq \frac{1}{2}$ for all $t \geq 0$, (iii)(b) is also satisfied. Finally, $\lambda(t) = \frac{1}{2}e^{0.1t} \geq 0$ for all $t \geq 0$. We have therefore found the solution.

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9.11.2 The current value Hamiltonian (with $\lambda_0 = 1$) is $H^c = x(2-u) + \lambda u x e^{-t}$, and the following conditions must be satisfied for $(x^*(t), u^*(t))$ to be optimal:

(i) $u = u^*(t)$ maximizes $H^c(t, x^*(t), u, \lambda(t)) = 2x^*(t) + x^*(t)(e^{-t}\lambda(t) - 1)u$ for $u \in [0, 1]$;

(ii) $\dot{\lambda}(t) - \lambda(t) = -\partial(H^c)^*/\partial x = -(2 - u^*(t)) - \lambda(t)u^*(t)e^{-t}$;

(iii) $\dot{x}^*(t) = u^*(t)x^*(t)e^{-t}, \ x^*(0) = 1$.

Since $\dot{x}^*(t) \geq 0$ and $x^*(0) = 1$, $x^*(t) \geq 1$ for all t. Looking at (i) we see that

$$u^*(t) = \begin{cases} 1 & \text{if } \lambda(t)e^{-t} > 1 \\ 0 & \text{if } \lambda(t)e^{-t} < 1 \end{cases}$$

We guess that $p(t) = \lambda(t)e^{-t} \to 0$ as $t \to \infty$. Then $\lambda(t)e^{-t} < 1$ on some maximal interval (t^*, ∞). On this interval $u^*(t) = 0$ and $\dot{\lambda}(t) - \lambda(t) = -2$, and so $\lambda(t) = Ce^t + 2$, or $\lambda(t)e^{-t} = C + 2e^{-t}$, which tends to 0 as $t \to \infty$ only if $C = 0$. Then $\lambda(t) = 2$. Note that $\lambda(t)e^{-t^*} = 2e^{-t^*} = 1$ when $t^* = \ln 2$, and we propose that $u^*(t) = 1$ on $[0, \ln 2]$, $u^*(t) = 0$ on $(\ln 2, \infty)$. Then $\dot{x}^*(t) = x^*(t)e^{-t}$ on $[0, \ln 2]$, $\dot{x}^*(t) = 0$ on $(\ln 2, \infty)$. On $[0, \ln 2]$, $\int dx^*(t)/x^*(t) = \int e^{-t}\,dt$, so $\ln x^*(t) = -e^{-t} + A$, and with $x^*(0) = 1$ this gives $x^*(t) = e^{1-e^{-t}}$. On (t^*, ∞) we have $x^*(t) = e^{1-e^{-\ln 2}} = e^{1-1/2} = e^{1/2}$.

On $[0, \ln 2]$ we have $\dot{\lambda}(t) + (e^{-t} - 1)\lambda(t) = -1$. We use formula (5.4.6) with $a(t) = e^{-t} - 1$ and $b(t) = -1$. Then $\int a(t)\,dt = \int (e^{-t} - 1)\,dt = -e^{-t} - t$ and

$$\lambda(t) = Ce^{e^{-t}+t} + e^{e^{-t}+t}\int e^{-e^{-t}-t}(-1)\,dt = Ce^{e^{-t}+t} - e^{e^{-t}+t}\int e^{-e^{-t}}e^{-t}\,dt = Ce^{e^{-t}+t} - e^t$$

because the last integral is obviously equal to $e^{-e^{-t}}$. Since $\lambda(\ln 2) = 2$ we find that $C = 2e^{-1/2}$. We have obtained the same answers as in the book, keeping in mind that $p(t) = \lambda(t)e^{-t}$.

It does not change the problem if $x(\infty) \geq 0$ is added as a restriction in the problem. Then (B) and (C) in Note 9.11.3 are trivially satisfied, and the expression in (A) reduces to $\lim_{t\to\infty} 2e^{-t}(0 - e^{1/2})$, which is clearly 0. The Arrow concavity condition holds in this problem and this yields optimality also in the infinite horizon case.

9.11.4 The current value Hamiltonian $H^c = (x - u) + \lambda u e^{-t}$ is concave in (x, u). The problem is not affected by introducing the requirement that $\lim_{t\to\infty} x^*(t) \geq 0$. So the following conditions are sufficient for optimality:

(i) $u = u^*(t)$ maximizes $x^*(t) + e^{-t}(\lambda(t) - e^t)u$ for $u \in [0, 1]$;

(ii) $\dot{\lambda}(t) - \lambda(t) = -(H^c)'_x(t, x^*, u^*, \lambda(t)) = -1$;

(iii) (a) $\lim_{t\to\infty} \lambda(t)e^{-t}(-x^*(t)) \geq 0$;

(b) There exists a number M s.t. $|\lambda(t)e^{-t}| \leq M$ for all $t \geq 0$;

(c) There exists a number t' s.t. $\lambda(t) \geq 0$ for all $t \geq t'$;

(iv) $\dot{x}^*(t) = u^*(t)e^{-t}, \ x^*(-1) = 0$.

From (ii) it follows that $\lambda(t) = Ae^t + 1$. We guess that $p(t) = e^{-t}\lambda(t) = e^{-t}(Ae^t + 1) = A + e^{-t} \to 0$ as $t \to \infty$, so $A = 0$. From (i) we see that $u^*(t) = 1$ if $e^t < 1$, and $u^*(t) = 0$ if $e^t > 1$. It follows that $u^*(t) = 1$ in $[-1, 0]$ and $u^*(t) = 0$ in $(0, \infty)$. Then from (iv), we get $x^*(t) = e - e^{-t}$ in $[-1, 0]$ and $x^*(t) = e - 1$ in $(0, \infty)$. The conditions in (iii) are obviously satisfied, so we have found the optimal solution. (The answer in the book is wrong.)

9.12

9.12.1 (a) The current value Hamiltonian $H^c = ax - \frac{1}{2}u^2 + \lambda(-bx + u)$ is concave in (x, u), so the following conditions are sufficient for optimality:

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(i) $(H^c)'_u(t, x^*(t), u^*(t), \lambda(t)) = -u^*(t) + \lambda(t) = 0$;
(ii) $\dot{\lambda}(t) - r\lambda(t) = -(H^c)'_x(t, x^*(t), u^*(t), \lambda(t)) = -a + b\lambda(t)$;
(iii) $\lim_{t\to\infty} \lambda(t)e^{-rt}(x(t) - x^*(t)) \geq 0$ for all admissible $x(t)$;
(iv) $\dot{x}^*(t) = -bx^*(t) + u^*(t),\ x^*(0) = 10$.

From (i) we get $u^*(t) = \lambda(t)$, and thus $\dot{x}^*(t) = -bx^*(t) + \lambda(t)$, and $x = x^*$ and λ must satisfy

$$\dot{x} = -bx + \lambda = F(x, \lambda), \qquad \dot{\lambda} = (b + r)\lambda - a = G(x, \lambda) \tag{$*$}$$

The equilibrium point is $(\bar{x}, \bar{\lambda}) = (a/b(b + r),\ a/(b + r)$.

(b) The Jacobian matrix of $(*)$ is $\begin{pmatrix} F'_x & F'_\lambda \\ G'_x & G'_\lambda \end{pmatrix} = \begin{pmatrix} -b & 1 \\ 0 & b+r \end{pmatrix}$, with eigenvalues $-b$ and $b+r$. Since they are real and of opposite signs, the equilibrium point is a saddle point. To prove sufficiency, we can restrict attention to admissible $x(t)$ satisfying $\lim_{t\to\infty} x(t)e^{-rt} \geq 0$, because if the limit is < 0, then the value of the criterion is $-\infty$. For such $x(t)$, condition (iii) is evidently satisfied.

(c) $V = \int_0^\infty [ax^*(t) - \frac{1}{2}(u^*(t))^2]e^{-rt}\,dt = \int_0^\infty ax_0e^{-(b+r)t}\,dt$ + terms that do not depend on x_0. But then $\partial V/\partial x_0 = \int_0^\infty ae^{-(b+r)t}\,dt = a/(b + r) = \lambda(0)$.

9.12.2 From the answer to Problem 9.9.1 with $x(0) = 1$, we find $x^* = Ae^{(1+\sqrt{2})t} + (1 - A)e^{(1-\sqrt{2})t}$. and $\lambda = A\sqrt{2}\,e^{(\sqrt{2}+1)t} - (1 - A)\sqrt{2}\,e^{(1-\sqrt{2})t}$. The adjoint variable $p(t) = \lambda(t)e^{-2t}$ tends to 0 as $t \to \infty$ if and only if $A = 0$. Then $x^* = e^{(1-\sqrt{2})t}$ and $\lambda(t) = -\sqrt{2}\,e^{(1-\sqrt{2})t}$ (with $p(t) = -\sqrt{2}\,e^{(-1-\sqrt{2})t}$). To prove sufficiency, note that if $\lim_{t\to\infty} x^2(t)e^{-2t} = \lim_{t\to\infty}(x(t)e^{-t})^2 \neq 0$, then the objective function is $-\infty$. We can therefore restrict attention to admissible solutions for which $\lim_{t\to\infty} x(t)e^{-t} = 0$. Then condition (iii) is satisfied and we have found the optimal solution.

9.12.3 (a) With $H^c = \ln C + \lambda(AK^\alpha - C)$, $\partial(H^c)^*/\partial C = 0$ implies $1/C^* - \lambda = 0$, or $C^*\lambda = 1$. Taking ln of each side and differentiating w.r.t. t yields $\dot{C}^*/C^* + \dot{\lambda}/\lambda = 0$. Also, $\dot{\lambda} - r\lambda = -\partial(H^c)^*/\partial K = -\lambda\alpha A(K^*)^{\alpha-1}$ or, equivalently, $\dot{\lambda}/\lambda = r - \alpha A(K^*)^{\alpha-1}$. It follows that if $K = K^*(t) > 0$ and $C = C^*(t) > 0$ solve the problem, then the second equation in $(*)$ holds. (The first equation is part of the problem.)

(b) The Jacobian matrix evaluated at (400, 40) is $J = \begin{pmatrix} 1/20 & -1 \\ -1/400 & 0 \end{pmatrix}$ and $|J| = -1/400 < 0$, so the equilibrium point is a saddle point.

(c) If $K_0 = 100$ and $T = \infty$ the solution curve converges towards the equilibrium point. For sufficient conditions, see Note 9.11.3.

9.12.4 (a) The current value Hamiltonian $H^c = -(x-1)^2 - \frac{1}{2}u^2 + \lambda(x-u)$ is concave in (x, u). Since $u \in \mathbb{R}$, the maximum condition reduces to (i) $(H^c)'_u(t, x^*, u^*, \lambda(t)) = -u^*(t) - \lambda = 0$. The differential equation for λ is (ii) $\dot{\lambda}(t) - \lambda(t) = -(H^c)'_x(t, x^*, u^*, \lambda(t)) = 2x^*(t) - 2 - \lambda$. Of course, $\dot{x}^*(t) = x^*(t) - u^*(t)$. It follows that the optimal pair $x^*(t), u^*(t))$ must satisfy

$$\begin{aligned} \dot{x} &= F(x, \lambda) = x - u = x + \lambda \\ \dot{\lambda} &= G(x, \lambda) = 2x - 2 \end{aligned} \tag{$*$}$$

The Jacobian matrix of $(*)$ is $\begin{pmatrix} F'_x & F'_\lambda \\ G'_x & G'_\lambda \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. The determinant is -2, so the equilibrium point $(1, -1)$ is a saddle point. The eigenvalues are -1 and 2.

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(b) Solving the first equation in $(*)$ for λ yields $\lambda = \dot{x} - x$, and then $\dot{\lambda} = \ddot{x} - \dot{x}$. Inserted into the second equation in $(*)$ yields $\ddot{x} - \dot{x} - 2x = -2$. The general solution of this equation is $x(t) = Ae^{-t} + Be^{2t} + 1$, and $x(0) = 1/2$ yields $B = -A - 1/2$, so

$$x(t) = Ae^{-t} - (A + 1/2)e^{2t} + 1$$

Using $\lambda = \dot{x} - x$, we find the corresponding solution for λ,

$$\lambda(t) = -2Ae^{-t} - (A + 1/2)e^{2t} - 1$$

If $A \neq -1/2$ we see that $x(t)$ and $\lambda(t)$ both diverge as $t \to \infty$. For $A = -1/2$, it follows that $x(t) = -(1/2)e^{-t} + 1 \to 1$ and $\lambda(t) = e^{-t} - 1 \to -1$ as $t \to \infty$. Note that from $x(t) = -(1/2)e^{-t} + 1$ and $\lambda(t) = e^{-t} - 1$ we find that $(x, \lambda) = (x(t), \lambda(t))$ satisfies $\lambda = -2x + 1$.

It remains to prove that condition (d) in Theorem 9.11.1 is satisfied. Note that the integrand in the objective function can be written $(-x^2 + 2x - 1 - \frac{1}{2}u^2)e^{-t}$, so we need only consider admissible $x(t)$ for which $\lim_{t\to\infty} x(t)e^{-t} \geq 0$, because if $\lim_{t\to\infty} x(t)e^{-t} < 0$ the objective function is $-\infty$.

10 Control Theory with Many Variables

10.1

10.1.3 To prove that g is concave, let $\mathbf{x}_1, \mathbf{x}_2, \lambda \in (0, 1)$ and choose $\mathbf{u}_1, \mathbf{u}_2$ in $U_{\mathbf{x}}$ such that $g(\mathbf{x}_1) = F(\mathbf{x}_1, \mathbf{u}_1)$, $g(\mathbf{x}_2) = F(\mathbf{x}_2, \mathbf{u}_2)$. Then $g(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq F(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, \lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2) \geq \lambda F(\mathbf{x}_1, \mathbf{u}_1) + (1-\lambda)F(\mathbf{x}_2, \mathbf{u}_2) = \lambda g(\mathbf{x}_1) + (1-\lambda)g(\mathbf{x}_2)$. Note that $\lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2 \in U_{\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2}$.

10.2

10.2.2 Suppose $T < 2/a$. If $t \in [0, T - 2/a]$, then $u^*(t) = 1$ and then $\dot{x}_1^*(t) = ax_1^*(t)$ with $x_1^*(0) = x_1^0$. If follows that $x_1^*(t) = x_1^0 e^{at}$. In particular, $x_1^*(T - 2/a) = x_1^0 e^{aT-2}$. Moreover, we have $\dot{x}_2^*(t) = a(1 - u^*(t))x_1^*(t) = 0$, and so $x_2^*(t) = x_2^0$.

If $t \in (T - 2/a, T]$, then $u^*(t) = 0$ and $\dot{x}_1^*(t) = 0$ with $x_1^*(T - 2/a) = x_1^0 e^{aT-2}$, so $x_1^*(t) = x_1^0 e^{aT-2}$. Moreover, $\dot{x}_2^*(t) = ax_1^0 e^{aT-2}$, so integration gives $x_2^*(t) = ax_1^0 e^{aT-2}t + B$, with the boundary condition $x_2^*(T - 2/a) = x_2^0$ determining B. In fact, we get $x_2^*(t) = x_2^0 + ax_1^0 e^{aT-2}(t - (T - 2/a))$.

10.2.3 (a) There are two state variables x_1 and x_2, so we introduce two adjoint variables p_1 and p_2. There are also two control variables u_1 and u_2. The Hamiltonian $H = \frac{1}{2}x_1 + \frac{1}{5}x_2 - u_1 - u_2 + p_1u_1 + p_2u_2$ is linear and hence concave. The following conditions are sufficient for the admissible quadruple $(x_1^*, x_2^*, u_1^*, u_2^*)$ to be optimal:

(i) $(u_1, u_2) = (u_1^*(t), u_2^*(t))$ maximizes $(p_1(t) - 1)u_1 + (p_2(t) - 1)u_2$ for $u_1 \in [0, 1]$, $u_2 \in [0, 1]$;

(ii) $\dot{p}_1(t) = -(H'_{x_1})^* = -1/2$, $p_1(T) = 0$, $\quad \dot{p}_2(t) = -(H'_{x_2})^* = -1/5$, $p_2(T) = 0$;

We see immediately from (ii) that $p_1(t) = \frac{1}{2}(T - t)$ and $p_2(t) = \frac{1}{5}(T - t)$. Note that $p_1(0) = \frac{1}{2}T > 1$ since $T > 5$. Thus $p_1(t)$ strictly decreases from a level higher than 1 at $t = 0$ to 0 at $t = T$. Looking at (i) we see that $u_1^*(t) = 1$ for $p_1(t) > 1$ and 0 for $p_1(t) < 1$. Since $p_1(t^*) = 1$ when $\frac{1}{2}(T - t^*) = 1$, we get $t^* = T - 2$. In the same way, $p_2(0) = \frac{1}{5}T > 1$ since $T > 5$. Thus $p_2(t)$ strictly decreases from

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a level higher than 1 at $t = 0$ to 0 at $t = T$. Looking at (i) we see that $u_2^*(t) = 1$ for $p(t) > 1$ and 0 for $p(t) < 1$. Since $p_2(t^{**}) = 1$ when $\frac{1}{5}(T - t^{**}) = 1$, we get $t^{**} = T - 5$. We conclude that

$$u_1^*(t) = \begin{cases} 1 & \text{if } t \in [0, T-2] \\ 0 & \text{if } t \in (T-2, T] \end{cases} \qquad u_2^*(t) = \begin{cases} 1 & \text{if } t \in [0, T-5] \\ 0 & \text{if } t \in (T-5, T] \end{cases}$$

The corresponding values for $(x_1^*(t), x_2^*(t))$ are easily worked out. See the answer in the book.

(b) The scrap value function is $S(x_1, x_2) = 3x_1 + 2x_2$. The Hamiltonian and the differential equations for p_1 and p_2 are the same as in (a). The transversality conditions are changed. In fact, according to Theorem 10.1.5 (C)(c'), we have $p_1(T) = S_1'(x_1^*(t), x_2(t)) = 3$ and $p_2(T) = S_2'(x_1^*(t), x_2(t)) = 2$. Then $p_1(t) = 3 + \frac{1}{2}(T - t)$ and $p_2(t) = 2 + \frac{1}{5}(T - t)$. For $t \in [0, T]$ we see that $p_1(t)$ and $p_2(t)$ are both greater than 1, and condition (i) in (a) implies that $u_1^*(t) = u_2^*(t) = 1$ for all $t \in [0, T]$, and thus $x_1^*(t) = x_2^*(t) = t$.

10.2.4 The Hamiltonian $H = x_2 + c(1 - u_1 - u_2) + p_1 a u_1 + p_2(a u_2 + b x_1)$ is linear and hence concave. The control region is $U = \{(u_1, u_2) : 0 \le u_1, 0 \le u_2, u_1 + u_2 \le 1\}$. The following conditions are sufficient for a quadruple $(x_1^*, x_2^*, u_1^*, u_2^*)$ to be admissible and optimal:

(i) $(u_1, u_2) = (u_1^*(t), u_2^*(t))$ maximizes $(ap_1(t) - c)u_1 + (ap_2(t) - c)$ for $(u_1, u_2) \in U$;

(ii) $\dot{p}_1(t) = -(H'_{x_1})^* = -bp_2(t)$, $p_1(T) = 0$, $\quad \dot{p}_2(t) = -(H'_{x_2})^* = -1$, $p_2(T) = 0$;

(iii) $\dot{x}_1^*(t) = au_1^*(t)$, $x_1^*(0) = x_1^0$, $\quad \dot{x}_2^*(t) = au_2^*(t) + bx_1^*(t)$, $x_2^*(0) = x_2^0$.

From (ii) we see that $p_2(t) = T - t$. Therefore $\dot{p}_1(t) = -b(T - t)$. Integrating and using $p_1(T) = 0$, we get $p_1(t) = \frac{1}{2}b(T - t)^2$. To find the optimal controls we must for each t in $[0, T]$ solve the problem

$$\max \varphi(u_1, u_2) = \max \Big\{ \big[\tfrac{1}{2}ab(T - t)^2 - c\big]u_1 + \big[a(T - t) - c\big]u_2 \quad \text{s.t.} \quad (u_1, u_2) \in U \Big\} \tag{$*$}$$

The control region U is the closed triangular region with corners at (0, 0), (1, 0), and (0, 1). Since φ is a linear function it will attain its maximum over U at one of those corners. (In some cases there may be more than one maximum point, but even then at least one corner will be a maximum point.) Note that $\varphi(0, 0) = 0$, $\varphi(1, 0) = \frac{1}{2}ab(T - t)^2 - c$, and $\varphi(0, 1) = a(T - t) - c$. In the chains of equivalences below it is understood that either the top inequality holds all the way or the bottom inequality holds all the way. We see that (with $t < T$)

$$\varphi(1,0) \gtrless \varphi(0,1) \iff \tfrac{1}{2}ab(T-t)^2 - c \gtrless a(T-t) - c \iff t \lessgtr T - 2/b$$

$$\varphi(0,1) \gtrless \varphi(0,0) \iff a(T-t) - c \gtrless 0 \iff t \lessgtr T - c/a$$

$$\varphi(1,0) \gtrless \varphi(0,0) \iff T - t \gtrless \sqrt{2c/ab} \iff t \lessgtr T - \sqrt{2c/ab}$$

Putting all this together (note that the assumption $T - c/a > T - 2/b$ in the problem implies $c/a < 2/b$, and then $c/a < \sqrt{2c/ab} < 2/b$), we find that a set of optimal controls are

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (1, 0) & \text{if } t \in [0, T - 2/b] \\ (0, 1) & \text{if } t \in (T - 2/b, T - c/a] \\ (0, 0) & \text{if } t \in (T - c/a, T] \end{cases} \tag{$**$}$$

(For values of t in the corresponding *open* intervals, the optimal values of u_1 and u_2 are uniquely determined; for $t = T - 2/b$, $t = T - c/a$, and $t = T$ we choose the values so that the control functions

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become left-continuous.) The corresponding values of $x_1^*(t)$ and $x_2^*(t)$ follow from (iii) and (∗), but the expressions on the interval $(T - c/a, T - 2/b)$ get very messy.

10.2.5 There are two state variables x_1 and x_2, so we introduce two adjoint variables p_1 and p_2. The Hamiltonian is $H(t, x_1, x_2, u, p_1, p_2) = x_1 - cx_2 + u^0 - u + p_1 u + p_2 b x_1$. (We have $p_0 = 1$, since $p_1(T) = p_2(T) = 0$.) The Hamiltonian is concave in (x_1, x_2, u), so according to Theorem 10.1.2 the admissible triple (x_1^*, x_2^*, u^*) is optimal provided for each t in $[0, T]$,

(i) $u = u^*(t)$ maximizes $H(t, x_1^*(t), x_2^*(t), u, p_1(t), p_2(t)) = x_1^*(t) - cx_2^*(t) + u^0 - u + p_1(t)u + p_2(t)bx_1^*(t) = \Gamma + p_2(t)bx_1^*(t) + (p_1(t) - 1)u$ for u in $[0, u^0]$ (where Γ does not depend on u).

(ii) $\dot{p}_1(t) = -\partial H^*/\partial x_1 = -1 - bp_2(t)$, $p_1(T) = 0$ and $\dot{p}_2(t) = -\partial H^*/\partial x_2 = c$, $p_2(T) = 0$

From (i) we see that $p_1(t) > 1 \Rightarrow u^*(t) = u^0$, $p_1(t) < 1 \Rightarrow u^*(t) = 0$. Moreover, from (ii), $p_2(t) = c(t - T)$, and $\dot{p}_1(t) = -1 - bp_2(t) = -1 - bc(t - T)$, which gives $p_1(t) = T - t - \frac{1}{2}bc(t - T)^2 = -\frac{1}{2}bct^2 + (bcT - 1)t + T - \frac{1}{2}bcT^2$, since $p_1(T) = 0$. Note that $p_1(t)$ is a quadratic polynomial in t, with $p_1(0) = T(1 - \frac{1}{2}bcT)$ and maximum at $t_1 = T - 1/bc$ (since $\dot{p}_1(t_1) = 0$). The maximum value of p_1 is $p_1(t_1) = 1/2bc$.

We restrict our attention to the main case $bcT > 2$ and $2bc < 1$. (The other cases are much simpler.) We then get $p_1(0) = T(1 - \frac{1}{2}bcT) < 0$, $p_1(t_1) = 1/2bc > 1$, $p_1(T) = 0$. The graph of p is shown in Figure 10.2.5.

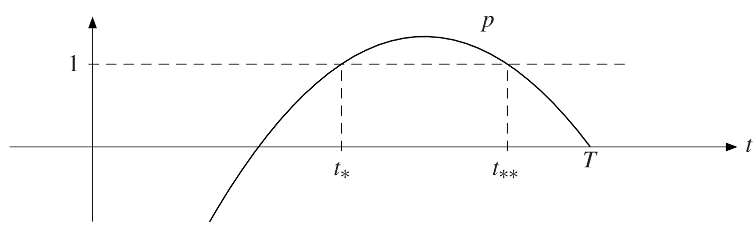

Figure 10.2.5

There will be two points t_* and t_{**}, such that $0 < t_* < t_{**} < T$ and $p_1(t_*) = p_1(t_{**}) = 1$. Now,

$$p_1(t) = \tfrac{1}{2}bct^2 - (bcT - 1)t - T + \tfrac{1}{2}bcT^2 + 1 = 0 \iff t = T - \frac{1}{bc} \pm \frac{1}{bc}\sqrt{1 - 2bc}$$

This gives

$$t_* = T - \frac{1}{bc} - \frac{1}{bc}\sqrt{1 - 2bc}\,, \qquad t_{**} = T - \frac{1}{bc} + \frac{1}{bc}\sqrt{1 - 2bc}$$

The optimal control u^* and the corresponding $x_1^*(t)$ are given by

$$u^*(t) = \begin{cases} 0 & \text{for } t \text{ in } [0, t_*], \\ 1 & \text{for } t \text{ in } (t_*, t_{**}]\,, \\ 0 & \text{for } t \text{ in } (t_{**}, T] \end{cases} \qquad x_1^*(t) = \begin{cases} x_1^0 & \text{for } t \text{ in } [0, t_*], \\ t - t_* + x_1^0 & \text{for } t \text{ in } (t_*, t_{**}] \\ t_{**} - t_* + x_1^0 & \text{for } t \text{ in } (t_{**}, T] \end{cases}$$

The expression for $x_2^*(t)$ is somewhat messy:

$$x_2^*(t) = \begin{cases} bx_1^0 t + x_2^0 & \text{for } t \text{ in } [0, t_*], \\ \frac{1}{2}bt^2 + b(x_1^0 - t_*)t + x_2^0 + \frac{1}{2}b(t_*)^2 & \text{for } t \text{ in } (t_*, t_{**}] \\ b(t_{**} - t_*)t + bx_1^0 t + \frac{1}{2}b[(t_*)^2 - (t_{**})^2] + x_2^0 & \text{for } t \text{ in } (t_{**}, T] \end{cases}$$

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10.2.7 There are two state variables and one control variable. We shall use Theorem 10.1.2. The Hamiltonian is $H(t, x, y, u, p_1, p_2) = x + (p_1 + p_2 - \frac{1}{2})u$. Suppose (x^*, y^*, u^*) is an admissible triple which solves the problem. Then:

(i) For each t in $[0, 2]$, $u = u^*(t)$ maximizes
$H(t, x^*(t), y^*(t), u, p_1(t), p_2(t)) = x^*(t) + (p_1(t) + p_2(t) - \frac{1}{2})u$ for u in $[0, 1]$;

(ii) $\dot{p}_1(t) = -\partial H^*/\partial x = -1, \quad \dot{p}_2(t) = -\partial H^*/\partial y = 0$;

(iii) (a) $p_1(2) = 0$,
(b) $p_2(2) \leq 0$, and $p_2(2) = 0$ if $y^*(2) < 1$;

(iv) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 1$, $\dot{y}^*(t) = u^*(t)$, $y^*(0) = 0$.

From (ii), $\dot{p}_1 = -1$, and since $p_1(2) = 0$, we have $p_1(t) = 2 - t$. Because $\dot{p}_2 = 0$, $p_2(t) = \bar{p}_2$, where $\bar{p}_2$ is a constant. Hence, $p_1(t) + p_2(t) - \frac{1}{2} = 2 - t + \bar{p}_2 - \frac{1}{2} = t^* - t$, where $t^* = 3/2 + \bar{p}_2$ is a constant. Condition (i) gives

$$u^*(t) = \begin{cases} 1 & \text{if } t < t^* \\ 0 & \text{if } t > t^* \end{cases}$$

If $t^* = 2$, then $u^*(t) \equiv 1$ and $y^*(t) \equiv t$, and then $y^*(2) > 1$, which is impossible.

If $t^* = 0$, then $u^*(t) \equiv 0$ and $y^*(t) \equiv 0$, which gives $y^*(2) = 0 < 1$. In this case $\bar{p}_2 = p_2(2) = 0$ because of (iii). Hence, $t^* = 3/2$, contradicting $t^* = 0$.

Thus we must have $0 < t^* < 2$, and then $x^*(t) - 1 = y^*(t) = t$ for $t \leq t^*$ and $x^*(t) - 1 = y^*(t) = t^*$ for $t > t^*$. In particular, $y^*(2) = t^* = 3/2 + \bar{p}_2$. It remains to determine $\bar{p}_2$ and t^*. Since $y^*(2) \leq 1$, we must have $3/2 + \bar{p}_2 \leq 1$, i.e. $\bar{p}_2 \leq -1/2 < 0$. From (iii), $y^*(2) = 1$, and so $\bar{p}_2 = -1/2$ and $t^* = 1$. The Hamiltonian is concave in (x, y, u) (in fact linear), so we have found a solution:

$$u^*(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}, \quad x^*(t) - 1 = y^*(t) = \begin{cases} t & \text{if } t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}, \quad p_1(t) = 2 - t, \quad p_2(t) = -\tfrac{1}{2}$$

10.3

10.3.1 The Hamiltonian $H = (x - u)e^{-rt} + p(t)ue^{-t}$ is concave in (x, u). If $(x^*(t), u^*(t))$ is optimal, then

(i) $u = u^*(t)$ maximizes $(e^{-t}p(t) - e^{-rt})u$ for $u \in [0, 1]$;

(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -e^{-rt}$;

(iii) $\dot{x}^*(t) = u^*(t)e^{-t}$, $x^*(0) = x_0 \geq 0$.

From (ii) we get $p(t) = (1/r)e^{-rt} + C$. There is no restriction on $x(t)$ as $t \to \infty$, so we guess that $p(t) \to 0$ as $t \to \infty$. Then we must have $C = 0$, so $p(t) = (1/r)e^{-rt}$. With this choice of $p(t)$, we see that $u = u^*(t)$ maximizes $(1/r)e^{-rt}(e^{-t} - r)u$ for $u \in [0, 1]$. Thus, $u^*(t) = 1$ if $e^{-t} > r$ and $u^*(t) = 0$ if $e^{-t} < r$. We have $e^{-t^*} = r$ when $t^* = -\ln r$, and then we see that (draw a picture!)

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, -\ln r] \\ 0 & \text{if } t \in (-\ln r, \infty) \end{cases}$$

From (iii) we find that $x^*(t) = x_0 + 1 - e^{-t}$ in $[0, -\ln r]$, while $x^*(t) = x_0 + 1 - r$ in $(-\ln r, \infty)$. Note that since $\dot{x} \geq 0$ for all $t \geq 0$ and $x(0) \geq 0$, we have $x(t) \geq x_0$ for all $t \geq 0$.

We have $p(t) = e^{-rt}/r \geq 0$, so according to Note 10.3.2 it remains only to verify (10.3.10a). In fact, we see that $\lim_{t\to\infty} p(t)(x_0 - x^*(t)) = \lim_{t\to\infty}(e^{-rt}/r)(x_0 - x_0 - 1 + r) = 0$.

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10.3.2 (a) The model is closely related to the model in Example 10.2.2. The Hamiltonian is $H = x_2 e^{-rt} + p_1 a u x_1 + p_2 a(1-u)x_1$. If $(x_1^*(t), x_2^*(t), u^*(t))$ is optimal, then

(i) $u = u^*(t)$ maximizes $a x_1^*(t)(p_1(t) - p_2(t))u$ for $u \in [0, 1]$;

(ii) $\dot{p}_1(t) = -a p_1(t) u^*(t) - a p_2(t)(1 - u^*(t))$, $\quad \dot{p}_2(t) = -e^{-rt}$;

(iii) $\dot{x}_1^*(t) = a u^*(t) x_1^*(t)$, $x_1^*(0) = x_1^0 > 0$, $\quad \dot{x}_2^*(t) = a(1-u^*(t))x_1^*(t)$, $x_2^*(0) = 0$.

From (ii) we get $p_2(t) = e^{-rt}/r + C$. There is no restriction on $x_2(t)$ as $t \to \infty$, so we guess that $p_2(t) \to 0$ as $t \to \infty$. Then we must have $C = 0$, so $p_2(t) = e^{-rt}/r$.

From (i) we see that

$$u^*(t) = \begin{cases} 1 & \text{if } p_1(t) > e^{-rt}/r \\ 0 & \text{if } p_1(t) < e^{-rt}/r \end{cases}$$

Suppose $p_1(0) > p_2(0) = 1/r$. Then $u^*(t) = 1$ to the immediate right of $t = 0$ and $\dot{p}_1(t) = -a p_1(t)$, so $p_1(t) = p_1(0)e^{-at} > (1/r)e^{-at} > (1/r)e^{-rt}$, since $r > a$, and we see that we must have $p_1(t) > p_2(t)$ for all $t \geq 0$. Then $u^*(t) \equiv 1$ and from (iii) we have $x_2^*(t) \equiv 0$, and the objective function is 0. This is not the optimal solution. (In the terminology of Example 10.2.2 the total discounted consumption is 0.) If $p_1(0) = p_2(0)$, then $\dot{p}_1(0) = -a p_2(0) = -a/r > -1 = \dot{p}_2(0)$, and again we see that $p_1(t) > p_2(t)$ for all $t \geq 0$. Suppose $p_1(0) < 1/r$. Then $u^*(t) = 0$ for t close to 0. Let us see if we can have $u^*(t) = 0$ for all t. Then $\dot{p}_1(t) = -a p_2(t) = -(a/r)e^{-rt}$ and so $p_1(t) = (a/r^2)e^{-rt} + D$. Again we must have $D = 0$ and then $p_1(t) = (a/r^2)e^{-rt} < (1/r)e^{-rt} = p_2(t)$. Finally, using (10.3.10(a)), we have $p_1(t)(x_1^0 - x_1^*(t)) = 0$ and $p_2(t)(0 - x_2^*(t)) = (1/r)e^{-rt}(-a x_1^0) \to 0$ as $t \to \infty$. As in Example 10.2.2 the Arrow condition is satisfied, so we have found the optimal solution. Note that using the interpretation in Example 10.2.2, in this case the discount factor r is so high that it is optimal with no further investment in the investment sector, which leads to consumption increasing at a constant rate.

(b) Choose the control $u(t) = b/a$, with $0 < r < b < a$. Then $u(t) \in [0, 1]$ and we seek corresponding admissible state variables. From $\dot{x}_1(t) = a(b/a)x_1(t) = b x_1(t)$, with $x_1(0) = x_1^0$, we find $x_1(t) = x_1^0 e^{bt}$. Then $\dot{x}_2(t) = (a-b)x_1^0 e^{bt}$, so $x_2(t) = (a-b)x_1^0(1/b)e^{bt} + C$. With $x_2(0) = 0$, we find that $C = -(a-b)(1/b)x_1^0$, so $x_2(t) = (a-b)(1/b)x_1^0(e^{bt} - 1)$. Then the objective function is $\int_0^\infty x_2(t)e^{-rt}\,dt = \int_0^\infty [(a-b)(1/b)x_1^0 e^{(b-r)t} - e^{-rt}]\,dt = \Big|_0^\infty [(a-b)(1/b)x_1^0[(1/(b-r)]e^{(b-r)t} + e^{-rt}/r]$. By using $0 < r < b < a$ we see that the integral diverges.

10.4

10.4.2 First we apply Theorem 10.4.1. The control region $U = [0, 1]$ is convex and the condition in Note 10.4.2 is satisfied because $|ux| \leq |x|$ since $|u| \leq 1$. The set $N(t, x) = \{((1-u)x^2 + \gamma, ux) : \gamma \leq 0, u \in [0.1]\}$ is convex (a rectangle) by the same argument as in Example 10.4.1. Thus there exists an optimal control.

The Hamiltonian is $H = (1-u)x^2 + pux = x^2 + ux(p - x)$, and if $(x^*(t), u^*(t))$ is optimal then:

(i) $u = u^*(t)$ maximizes $x^*(t)(p(t) - x^*(t))u$ for $u \in [0, 1]$;

(ii) $\dot{p}(t) = -(H'_x)^* = -2(1 - u^*(t))x^*(t) - p(t)u^*(t)$, $\quad p(1) = 0$;

(iii) $\dot{x}^*(t) = u^*(t)x^*(t)$, $\quad x^*(0) = x_0 > 0$.

From (iii) we see that $\dot{x}^*(t) \geq 0$ so $x^*(t) \geq x_0 > 0$. Thus (i) implies that

$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > x^*(t) \\ 0 & \text{if } p(t) < x^*(t) \end{cases}$$

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We claim that $p(t)$ is strictly decreasing. In fact, for those t where $p(t) > x^*(t)$ we have $u^*(t) = 1$ and $\dot{p}(t) = -p(t) < 0$. For those t where $p(t) < x^*(t)$ we have $u^*(t) = 0$ and $\dot{p}(t) = -2x^*(t) < 0$. Since $x^*(t)$ is increasing from the level $x_0 > 0$ and $p(t)$ is strictly decreasing and is 0 at $t = 1$, we have to have $p(t) < x^*(t)$ in some interval $(t^*, 1]$. Then $u^*(t) = 0$ in this interval.

If $t^* = 0$, then $u^*(t) \equiv 0$, $x^*(t) \equiv x_0$, and $\dot{p}(t) \equiv -2x_0$. Since $p(1) = 0$ we get $p(t) = 2x_0(1-t)$. But then $p(0) = 2x_0$, contradicting $p(0) \le x(0) = x_0$. With $t^* > 0$,

$$u^*(t) = \begin{cases} 1 & \text{if } t < t^* \\ 0 & \text{if } t > t^* \end{cases} \qquad \text{and} \qquad x^*(t) = \begin{cases} x_0 e^t & \text{if } t < t^* \\ x_0 e^{t^*} & \text{if } t > t^* \end{cases}$$

In $(t^*, 1]$ we have $\dot{p}(t) = -2x^*(t) = -2x_0e^{t^*}$ and $p(1) = 0$, so $p(t) = 2x_0e^{t^*}(1-t)$. But at t^* we have $p(t^*) = x^*(t^*)$, so $2x_0e^{t^*}(1-t^*) = x_0e^{t^*}$, from which it follows that $t^* = 1/2$. We find that $p(t) = x_0e^{1-t}$ in $[0, 1/2]$. We have found the optimal solution.

10.4.3 The Hamiltonian is $H = p_0x^2 + p(1-u^2)$. According to the maximum principle, if $(x^*(t), u^*(t))$ is an optimal pair, there exist a continuous and piecewise differentiable function $p(t)$ and a constant p_0, either 0 or 1, such that $(p_0, p(t)) \neq (0, 0)$ and

(i) $u = u^*(t)$ maximizes $p_0(x^*)^2 + p(t)(1-u^2)$ for $u \in [-1, 2]$;
(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -2p_0x^*(t)$;
(iii) $\dot{x}^*(t) = 1 - (u^*(t))^2$, $x^*(0) = x^*(1) = 4$.

Suppose $p_0 = 0$. Then from (ii), $p(t) = \bar{p} \neq 0$, and $u^*(t)$ maximizes $-\bar{p}u^2$ for $u \in [-1, 2]$. If $\bar{p} > 0$, then $u^*(t) \equiv 0$ and $\dot{x}^*(t) \equiv 1$, so with $x^*(0) = 4$, $x^*(t) = t + 4$, and then $x^*(1) \neq 4$. On the other hand, if $\bar{p} < 0$, then $u^*(t) \equiv 2$ and $\dot{x}^*(t) \equiv -3$, so with $x^*(0) = 4$, $x^*(t) = -3t + 4$, and then $x^*(1) \neq 4$. Thus, assuming $p_0 = 0$ leads to contradictions, so $p_0 = 1$.

An optimal control $u^*(t)$ must maximize $p(t)(1-u^2)$ for $u \in [-1, 2]$. Then we see that if $p(t) > 0$, one should use $u^*(t) = 0$, and if $p(t) < 0$, then one should use $u^*(t) = 2$. (In neither case should one use $u^*(t) = -1$!) From (iii) and $u \in [-1, 2]$ we see that $\dot{x}^*(t)$ is never less than -3. Since $x^*(0) = 4$, it means that $x^*(t)$ is always ≥ 1. (Formally, $x^*(t) - x^*(0) = \int_0^t \dot{x}^*(\tau)\,d\tau \ge \int_0^t (-3)\,d\tau = -3t$, so $x^*(t) \ge x^*(0) - 3t = 4 - 3t$, which is ≥ 1 in $[0, 1]$.) From (ii) it follows that $\dot{p}(t) = -2x^*(t) < 0$, so $p(t)$ is strictly decreasing.

Suppose $p(t) < 0$ for all $t \in [0, 1]$. Then $u^*(t) \equiv 2$ and we get a contradiction to $x^*(1) = 4$. If $p(t) > 0$ for all $t \in [0, 1]$, then $u^*(t) \equiv 0$ and again we get a contradiction to $x^*(1) = 4$. We conclude that the strictly decreasing function $p(t)$ is > 0 in some interval $[0, t^*]$ and < 0 in $(t^*, 1]$. Then

$$u^*(t) = \begin{cases} 0 & \text{if } t \in [0, t^*] \\ 2 & \text{if } t \in (t^*, 1] \end{cases}$$

In $[0, t^*]$ we have $\dot{x}^*(t) = 1$ and since $x^*(0) = 4$, $x^*(t) = t + 4$. In $(t^*, 1]$ we have $\dot{x}^*(t) = -3$, and since $x^*(1) = 4$, we have $x^*(t) = -3t + 7$. Now $p(t)$ is continuous at t^*, so $t^* + 4 = -3t^* + 7$, so $t^* = 3/4$.

It remains to find $p(t)$. On $[0, 3/4]$ we have $\dot{p}(t) = -2x^*(t) = -2t - 8$, and so $p(t) = -t^2 - 8t + C$. Since $p(3/4) = 0$, $C = 105/16$. In $(3/4, 1]$ we have $\dot{p}(t) = -2x^*(t) = 6t - 14$, and so $p(t) = 3t^2 - 14t + D$. Since $p(3/4) = 0$, $D = 141/16$. The answer is summed up in the answer in the book. (Note the misprint in line 3 of the answer to Problem 10.4.3 in the book: ... When $u = 2$, $\dot{x} = -3$ ...)

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10.6

10.6.1 (a) $H = -\frac{1}{2}u^2 - x - pu$ and $\mathcal{L} = H + q(x - u)$. H is concave in (x, u) and $h(t, x, u) = x - u$ is linear and therefore quasiconcave. The conditions in Theorem 10.6.1 are

(i) $\partial\mathcal{L}^*/\partial u = -u^*(t) - p(t) - q(t) = 0$;
(ii) $q(t) \geq 0$ $(= 0$ if $x^*(t) > u^*(t))$;
(iii) $\dot{p}(t) = 1 - q(t)$, with $p(2) = 0$.

(b) In $[0, t^*]$ we have $x^*(t) = u^*(t)$, so $\dot{x}^*(t) = -u^*(t) = -x^*(t)$, and with $x^*(0) = 1$ we get $x^*(t) = u^*(t) = e^{-t}$. From (i) we have $q(t) = -u^*(t) - p(t) = -e^{-t} - p(t)$, so (iii) gives $\dot{p}(t) = 1 - q(t) = 1 + e^{-t} + p(t)$. This linear differential equation has the solution $p(t) = Ae^t - 1 - \frac{1}{2}e^{-t}$. From the argument in the problem, $q(t^{*-}) = 0$. Thus $q(t^{*-}) = -e^{-t^*} - p(t^*) = 0$, so $p(t^*) = -e^{-t^*} = Ae^{t^*} - 1 - \frac{1}{2}e^{-t^*}$, or $Ae^{t^*} = 1 - \frac{1}{2}e^{-t^*}$.

In $(t^*, 1]$ we have $q(t) = 0$ and $\dot{p}(t) = 1$, and since $p(2) = 0$, we get $p(t) = t - 2$. Then from (i), $u^*(t) = -p(t) = 2 - t$, and so $\dot{x}^*(t) = t - 2$ and then $x^*(t) = \frac{1}{2}t^2 - 2t + B$. In particular, since $x^*(t)$ is continuous at t^*, $x^*(t^*) = \frac{1}{2}(t^*)^2 - 2t^* + B = e^{-t^*}$. This gives $B = e^{-t^*} - \frac{1}{2}(t^*)^2 + 2t^*$. Finally, since $p(t)$ is continuous at t^*, we have $t^* - 2 = Ae^{t^*} - 1 - \frac{1}{2}e^{-t^*} = 1 - \frac{1}{2}e^{-t^*} - 1 - \frac{1}{2}e^{-t^*} = -e^{-t^*}$, so $e^{-t^*} = 2 - t^*$. By using this relationship you will see that the values of A and B are the same as in the answer in the book.

10.6.2 This is a problem of the form (10.6.1)–(10.6.4). The Lagrangian (10.6.5) is here $\mathcal{L} = H + q(x - u) = x - \frac{1}{2}u^2 + pu + q(x - u)$. Note that the Hamiltonian $H = x - \frac{1}{2}x^2 + pu$ is concave (in fact linear) in (x, u) and that $h = x - u$ is quasiconcave (in fact linear) in (x, u). According to Theorem 10.6.1, the following conditions are sufficient for $(x^*(t), u^*(t))$ to be optimal: There exist a continuous function $p(t)$ and a piecewise continuous function $q(t)$ such that

(i) $\partial\mathcal{L}^*/\partial u = -u^*(t) + p(t) - q(t) = 0$;
(ii) $q(t) \geq 0$, with $q(t) = 0$ if $x^*(t) > u^*(t)$;
(iii) $\dot{p}(t) = -\partial\mathcal{L}^*/\partial x = -1 - q(t), \quad p(2) = 0$;
(iv) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 1$.

We guess that $u^*(t) = x^*(t)$ on some interval $[0, t^*]$, and then (iv) gives $\dot{x}^*(t) = x^*(t)$, with $x^*(0) = 1$, so $x^*(t) = e^t = u^*(t)$. Then from (i) we have $q(t) = p(t) - e^t$, and (iii) gives $\dot{p}(t) = -1 - q(t) = -1 - p(t) + e^t$, or $\dot{p}(t) + p(t) = -1 + e^t$. The solution of this linear differential equation is $p(t) = Be^{-t} + \frac{1}{2}e^t - 1$, and then $q(t) = Be^{-t} + \frac{1}{2}e^t - 1 - e^t = Be^{-t} - \frac{1}{2}e^t - 1$.

On $(t^*, 2]$ we guess that $x^*(t) > u^*(t)$. Then from (ii) we have $q(t) = 0$, and so (iii) gives $p(t) = 2 - t$. Then from (i) $u^*(t) = p(t) - q(t) = 2 - t$, and (iv) gives $\dot{x}^*(t) = 2 - t$, so $x^*(t) = -\frac{1}{2}t^2 + 2t + A$. Since $x^*(t)$ is continuous at t^*, we have $-\frac{1}{2}(t^*)^2 + 2t^* + A = e^{t^*}$, so $A = e^{t^*} + \frac{1}{2}(t^*)^2 - 2t^*$. Since $p(t)$ is also continuous at t^*, we get $Be^{-t^*} + \frac{1}{2}e^{t^*} - 1 = 2 - t^*$, so $Be^{-t^*} = -\frac{1}{2}e^{t^*} + 3 - t^*$. Finally, since $q(t^{*-}) = 0$ (see the argument in the previous problem), we get $Be^{-t^*} = \frac{1}{2}e^{t^*} + 1$. From the last two equalities we get $e^{t^*} = 2 - t^*$, i.e. $t^* \approx 0.44$. To confirm that all the conditions (i)–(iv) are satisfied, we should verify that $q(t) = (\frac{1}{2}e^{2t^*} + e^{t^*})e^{-t} - \frac{1}{2}e^t - 1$ is nonnegative in $[0, t^*]$. This is the case because we find that $\dot{q}(t) < 0$ and $q(t^*) = 0$. The solution is summed up in the answer in the book.

10.6.3 Note that there are two constraints, $h_1 = u - c \geq 0$ and $h_2 = ax - u \geq 0$. The Lagrangian is $\mathcal{L} = H + q_1(u - c) + q_2(ax - u) = u + p(ax - u) + q_1(u - c) + q_2(ax - u)$. The Hamiltonian is concave (in fact linear) in (x, u) and h_1 and h_2 are quasiconcave (in fact linear) in (x, u). According to Theorem 10.6.1, the following conditions are sufficient for an admissible pair $(x^*(t), u^*(t))$ to be optimal: There exist a continuous function $p(t)$ and piecewise continuous functions $q_1(t)$ and $q_2(t)$ such that

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(i) $\partial \mathcal{L}^*/\partial u = 1 - p(t) + q_1(t) - q_2(t) = 0$;
(ii) $q_1(t) \geq 0$, with $q_1(t) = 0$ if $u^*(t) > c$;
(iii) $q_2(t) \geq 0$, with $q_2(t) = 0$ if $ax^*(t) > u^*(t)$;
(iv) $\dot{p}(t) = -\partial \mathcal{L}^*/\partial x = -ap(t) - aq_2(t)$;
(v) $p(T) \geq 0$ with $p(T) = 0$ if $x^*(T) > x_T$;
(vi) $\dot{x}^*(t) = ax^*(t) - u^*(t)$, $x^*(0) = x^0$, $x^*(T) \geq x_T$;
(vii) $c \leq u^*(t) \leq ax^*(t)$

In the similar model in Example 10.6.2 it was optimal to start out with $u^*(t) = c$ in an initial interval, and then keep $x^*(t)$ constant, so let us try the same here:

$$u^*(t) = \begin{cases} c & \text{if } t \in [0, t'] \\ ax^*(t) & \text{if } t \in (t', T] \end{cases}, \quad \text{then} \quad x^*(t) = \begin{cases} (x^0 - c/a)e^{at} + c/a & \text{if } t \in [0, t'] \\ (x^0 - c/a)e^{at'} + c/a & \text{if } t \in (t', T] \end{cases} \tag{viii}$$

In the interval $[0, t']$ we have $u^*(t) < ax^*(t)$ because $c < (ax^0 - c)e^{at} + c$, and then (iii) gives $q_2(t) = 0$, so we have $\dot{p}(t) = -ap(t)$. In the interval $(t', T]$ we have $u^*(t) = ax^*(t) = (ax^0 - c)e^{at'} + c > c$, so according to (ii), $q_1(t) = 0$. Then (i) gives $q_2(t) = 1 - p(t)$, which inserted into (iv) gives $\dot{p}(t) = -a$. We claim moreover that $p(t')$ must be 1. In fact from (i), since $q_2(t'^-) = 0$, we have $1 - p(t'^-) = -q_1(t'^-) \leq 0$ and since $q_1(t'^+) = 0$, we have $1 - p(t'^+) = q_2(t'^+) \geq 0$. Because $p(t)$ is continuous, $p(t') = 1$. Then we get

$$\dot{p}(t) = \begin{cases} -ap & \text{if } t \in [0, t'] \\ -a & \text{if } t \in (t', T] \end{cases} \quad \text{and} \quad p(t') = 1 \quad \Rightarrow \quad p(t) = \begin{cases} e^{-a(t-t')} & \text{if } t \in [0, t'] \\ a(t' - t) + 1 & \text{if } t \in (t', T] \end{cases} \tag{ix}$$

The corresponding values of $q_1(t)$ and $q_2(t)$ are

$$q_1(t) = \begin{cases} e^{-a(t-t')} - 1 & \text{if } t \in [0, t'] \\ 0 & \text{if } t \in (t', T] \end{cases}, \quad q_2(t) = \begin{cases} 0 & \text{if } t \in [0, t'] \\ a(t - t') & \text{if } t \in (t', T] \end{cases} \tag{x}$$

Case A: $x^*(T) > x_T$. Then $p(T) = 0$ and thus $a(t' - T) + 1 = 0$, so $t' = t_A$, where $t_A = T - 1/a$. The optimal solution is given by (viii), $p(t)$ by (ix), and $q_1(t)$ and $q_2(t)$ by (x), all with t' replaced by $T - 1/a$. It is a useful exercise to check that all the conditions in (i)–(vii) are satisfied. In particular, check that $q_1(t)$ and $q_2(t)$ are both ≥ 0. We must also check that $x^*(T) > x_T$. We see from (viii) that this is equivalent to $(x^0 - c/a)e^{a(T-1/a)} + c/a > x_T$, or $t_A > t_B$, with t_B defined in (xi) below.

Case B: $x^*(T) = x_T$. Then from (viii) we get $(x^0 - c/a)e^{at'} + c/a = x_T$, which solved for t' gives $t' = t_B$, where

$$t_B = \frac{1}{a} \ln \left(\frac{x_T - c/a}{x^0 - c/a} \right) \tag{xi}$$

Note that from (v) we need to have $p(T) \geq 0$. The formula for $p(t)$ in (ix) gives $a(t_b - T) + 1 \geq 0$, or $t_B \geq t_A$. The solution with $t' = t_B$ is valid if the last inequality is satisfied.

The final conclusion is given in the answer in the book. (In line 2 of the answer to Problem 10.6.3, replace x_* by x^* and t^* by t'.)

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10.6.4 The Lagrangian is $\mathcal{L} = H + q_1(1-u) + q_2(1+u) + q_3(2-x-u) = x + p(x+u) + q_1(1-u) + q_2(1+u) + q_3(2-x-u)$. The Hamiltonian $H = x + p(x+u)$ is concave (in fact linear) in (x, u) and $h_1 = 1-u$, $h_2 = 1+u$, and $h_3 = 2-x-u$ are quasiconcave (in fact linear) in (x, u). According to Theorem 10.6.1, the following conditions are sufficient for $(x^*(t), u^*(t))$ to be an optimal pair: there exist a continuous function $p(t)$ and piecewise continuous functions $q_1(t)$, $q_2(t)$, and $q_3(t)$, such that

(i) $\partial\mathcal{L}^*/\partial u = p(t) - q_1(t) + q_2(t) - q_3(t) = 0$;
(ii) $q_1(t) \geq 0$, with $q_1(t) = 0$ if $u^*(t) < 1$;
(iii) $q_2(t) \geq 0$, with $q_2(t) = 0$ if $u^*(t) > -1$;
(iv) $q_3(t) \geq 0$, with $q_3(t) = 0$ if $u^*(t) + x^*(t) < 2$;
(v) $\dot{p}(t) = -\partial\mathcal{L}^*/\partial x = -1 - p(t) + q_3(t)$, $p(1) = 0$;
(vi) $\dot{x}^*(t) = x^*(t) + u^*(t)$, $x^*(0) = 0$, i.e. (x^*, u^*) is admissible.

If we disregard the constraint $x + u \leq 2$, this is the problem solved in Example 9.4.1, whose solution was $(x(t), u(t)) = (e^t - 1, 1)$. Note that in this case $x(t) + u(t) = e^t \leq 2$ as long as $t \leq \ln 2$. We guess that this is the optimal solution on the interval $[0, \ln 2]$ in the present problem too. In fact, we will try to guess the optimal solution and then verify its optimality by checking that all the conditions (i)–(vi) are satisfied. So we start out with $(x^*(t), u^*(t)) = (e^t - 1, 1)$ on $[0, \ln 2]$. At $t = \ln 2$ we have $x^*(\ln 2) = e^{\ln 2} - 1 = 1$, and, looking at the objective function, for $t > \ln 2$ it seems optimal to increase $x(t)$ as fast as the constraint $x + u \leq 2$ allows, i.e. putting $\dot{x}^*(t) = u^*(t) + x^*(t) = 2$, as long as the h_1 and h_2 constraints are not violated. Now, with $\dot{x}^*(t) = 2$ on $[\ln 2, 1]$, and $x^*(\ln 2) = 1$, we get $x^*(t) = 2t + 1 - 2\ln 2$. Then $u^*(t) = 2 - x^*(t) = 1 + 2\ln 2 - 2t$, and it is easy to verify that $u^*(t) = 1 + 2\ln 2 - 2t$ takes values in $(-1, 1)$ when $t \in (\ln 2, 1]$. The suggestion we have for an optimal solution is therefore: In $[0, \ln 2]$, $(x^*(t), u^*(t)) = (e^t - 1, 1)$, in $(\ln 2, 1]$, $(x^*(t), u^*(t)) = (2t + 1 - 2\ln 2, 1 + 2\ln 2 - 2t)$.

We know that the suggested solution is admissible. It remains to find appropriate multipliers satisfying (i)–(v).

In the interval $(\ln 2, 1]$, $u^*(t) \in (-1, 1)$, so from (ii) and (iii), $q_1(t) = q_2(t) = 0$. Then (i) gives $p(t) = q_3(t)$ and from (v), $\dot{p}(t) = -1$ with $p(1) = 0$, so $p(t) = 1 - t$. In particular, $p(\ln 2) = 1 - \ln 2$.

In the interval $[0, \ln 2)$, $u^*(t) = 1 > -1$ and $x^*(t) + u^*(t) = e^t < 2$. Then from (iii) and (iv), $q_2(t) = q_3(t) = 0$. Then (v) gives $\dot{p}(t) = -1 - p(t)$. Solving the linear differential equation on $[0, \ln 2]$ with $p(\ln 2) = 1 - \ln 2$ gives $p(t) = (4 - 2\ln 2)e^{-t} - 1$. Then from (i), $q_1(t) = p(t)$. The complete suggestion for an optimal solution is therefore:

	$u^*(t)$	$x^*(t)$	$p(t)$	$q_1(t)$	$q_2(t)$	$q_3(t)$
$t \in [0, \ln 2]$	$e^t - 1$	1	$(4 - 2\ln 2)e^{-t} - 1$	$(4 - 2\ln 2)e^{-t} - 1$	0	0
$t \in (\ln 2, 1]$	$2t + 1 - 2\ln 2$	$1 + 2\ln 2 - 2t$	$1 - t$	0	0	$1 - t$

Having checked that $(x^*(t), u^*(t))$ satisfies all the conditions (i)–(vi), we conclude that $(x^*(t), u^*(t))$ is optimal. Note that $q_1(t)$ and $q_3(t)$ have jump discontinuities at $t = \ln 2$.

10.7

10.7.1 We maximize $\int_0^5 (-u - x)\,dt$, so the Lagrangian is $\mathcal{L} = H + qx = -u - x + p(u - t) + qx$. Here H is concave in (x, u) and $h(t, x) = x$ is quasiconcave, so by Theorem 10.7.1, the conditions (i)–(vi) below are sufficient for $(x^*(t), u^*(t))$ to be optimal:

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(i) $u = u^*(t)$ maximizes $-u - x^*(t) + p(t)(u - t) = -x^*(t) - tp(t) + u(p(t) - 1)$ for $u \geq 0$;
(ii) $q(t) \geq 0$, with $q(t) = 0$ if $x^*(t) > 0$;
(iii) $\dot{p}(t) = -\partial \mathcal{L}^*/\partial x = 1 - q(t)$, $p(5) = 0$;
(iv) $p(5^-) - p(5) = \beta$;
(v) $\beta \geq 0$, with $\beta = 0$ if $x^*(5) > 0$;
(vi) $\dot{x}^*(t) = u^*(t) - t$, $x^*(0) = 1$.

Since we want to keep $x^*(t)$ down, we put $u^*(t) = 0$ in some interval $[0, t^*]$. Then $\dot{x}^*(t) = -t$ with $x^*(0) = 1$, so $x^*(t) = -\frac{1}{2}t^2 + 1$. We see that $x^*(t)$ is decreasing and is 0 at $t^* = \sqrt{2}$. In $[0, \sqrt{2})$ we have by (ii), $q(t) = 0$. Then (iii) gives $\dot{p}(t) = 1$, and thus $p(t) = t + A$, for some constant A.

In order still to keep $x^*(t)$ down, we try $u^*(t) = t$ in $(\sqrt{2}, 5]$. Then $\dot{x}^*(t) = 0$ and thus $x^*(t) = x^*(\sqrt{2}) = 0$. For $u^*(t) = t$ to be the maximizer in (i) one has to have $p(t) = 1$, in particular $p(5^-) = 1$. Since $p(t)$ is continuous at $t = \sqrt{2}$, we have $\sqrt{2} + A = 1$, so $A = 1 - \sqrt{2}$. From (iii) we get $q(t) = 1$. Finally, since $p(5) = 0$, (iv) gives $\beta = 1$. Now all the conditions (i)–(vi) are satisfied, so $(x^*(t), u^*(t))$ is optimal.

10.7.2 The Lagrangian is $\mathcal{L} = H + qx = 1 - x + pu + qx$. Here H is concave in (x, u) and $h(t, x) = x$ is quasiconcave, and the conditions (i)–(vi) below are therefore sufficient for $(x^*(t), u^*(t))$ to be optimal:

(i) $u = u^*(t)$ maximizes $1 - x^*(t) + p(t)u$ for $u \in [-1, 1]$;
(ii) $q(t) \geq 0$, with $q(t) = 0$ if $x^*(t) > 0$;
(iii) $\dot{p}(t) = -\partial \mathcal{L}^*/\partial x = 1 - q(t)$, $p(2) = 0$;
(iv) $p(2^-) - p(2) = \beta$;
(v) $\beta \geq 0$, with $\beta = 0$ if $x^*(2) > 0$;
(vi) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 1$.

We start by putting $u^*(t) = -1$. Then $x^*(t) = 1 - t$ is decreasing and is 0 at $t^* = 1$. In $[0, 1)$ we have by (ii), $q(t) = 0$. Then (iii) gives $\dot{p}(t) = 1$, and thus $p(t) = t + A$, for some constant A.

In order still to keep $x^*(t)$ down put $u^*(t) = 0$ in $(1, 2]$. Then with $x^*(1) = 0$ we get $x^*(t) = 0$. For $u^*(t) = 0$ to be the maximizer in (i) for $t \in (1, 2)$, one has to have $p(t) = 0$, in particular $p(2^-) = 0$. Then since $p(2) = 0$, we get from (iv) that $\beta = 0$. Since $p(t)$ is continuous at $t = 1$, $p(1) = 1 + A = 0$, so $A = -1$. Finally, from (iii) we get $q(t) = 1$. Now all the conditions (i)–(vi) are satisfied, so the optimal solution is:

$$(x^*(t), u^*(t), p(t)) = \begin{cases} (1 - t, -1, t - 1) & \text{if } t \in [0, 1] \\ (0, 0, 0) & \text{if } t \in (1, 2] \end{cases}, \qquad q(t) = \begin{cases} 0 & \text{if } t \in [0, 1] \\ 1 & \text{if } t \in (1, 2] \end{cases}$$

with $\beta = 0$.

10.7.3 The Lagrangian is $\mathcal{L} = H + qx = -u^2 - x + pu + qx$. Here H is concave in (x, u) and $h(t, x) = x$ is quasiconcave, so the conditions (i)–(vi) below are therefore sufficient for $(x^*(t), u^*(t))$ to be optimal:

(i) $u = u^*(t)$ maximizes $-x^*(t) + p(t)u - u^2$ for $u \in \mathbb{R}$;
(ii) $q(t) \geq 0$, with $q(t) = 0$ if $x^*(t) > 0$;
(iii) $\dot{p}(t) = -\partial \mathcal{L}^*/\partial x = 1 - q(t)$, $p(10) = 0$;
(iv) $p(10^-) - p(10) = \beta$;
(v) $\beta \geq 0$, with $\beta = 0$ if $x^*(10) > 0$;
(vi) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 1$.

Since H is concave in u and $u \in \mathbb{R}$, (i) is equivalent to $(H'_u)^* = p(t) - 2u^*(t) = 0$, so $u^*(t) = \frac{1}{2}p(t)$. We have $x^*(0) = 1$. Let $[0, t^*]$ be the maximal interval where $x^*(t) > 0$. Then from (ii), $q(t) = 0$,

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and (iii) gives $\dot{p}(t) = 1$, and thus $p(t) = t + A$. Then $u^*(t) = \frac{1}{2}(t + A)$ and $\dot{x}^*(t) = \frac{1}{2}(t + A)$, so $x^*(t) = \frac{1}{4}(t + A)^2 + B$. Since $x^*(0) = 1$, $B = -\frac{1}{4}A^2 + 1$. If $t^* = 10$, then $p(t) = t - 10$ since $p(10) = 0$, and $u^*(t) = \frac{1}{2}(t - 10) < 0$ for $t < 10$, contradicting $u^*(t) \geq 0$. Thus $t^* < 10$ and $x^*(t^*) = 0$, and so $x^*(t^*) = \frac{1}{4}(t^* + A)^2 - \frac{1}{4}A^2 + 1 = 0$. The function $x^*(t)$ must have a minimum at t^* so $\dot{x}^*(t^*) = \frac{1}{2}(t^* + A) = 0$, and so $A = -t^* < 0$. Hence $x^*(t^*) = 1 - \frac{1}{4}A^2 = 0$, so $A = -2$. With $p(t) = t - 2$ we have $u^*(t) = \frac{1}{2}(t - 2)$, and $x^*(t) = \frac{1}{4}(t - 2)^2$ in $[0, 2]$.

Looking at the objective function, when $x^*(t)$ has become 0, it is obvious that we need to keep $u^*(t) = 0$ on $(2, 10]$. So $u^*(t) = x^*(t) = 0$, and then $p(t) = 0$. Then from (iii), $q(t) = 1$ and (iv) gives $\beta = p(10^-) = 0$ since $p(10) = 0$. Now all the conditions (i)–(vi) are satisfied, so the optimal solution is

$$(x^*(t), u^*(t), p(t)) = \begin{cases} (\frac{1}{4}(t-2)^2, \frac{1}{2}(t-2), t-2) & \text{if } t \in [0, 2] \\ (0, 0, 0) & \text{if } t \in (2, 10] \end{cases}, \quad q(t) = \begin{cases} 0 & \text{if } t \in [0, 2] \\ 1 & \text{if } t \in (2, 10] \end{cases}$$

with $\beta = 0$.

10.7.4 (a) The Hamiltonian $H = (4 - t)u + pu$ is concave in (x, u), so the following conditions are sufficient for $(x^*(t), u^*(t))$ to be optimal:

(i) $u = u^*(t)$ maximizes $\big(p(t) - (t - 4)\big)u$ for $u \in [0, 2]$;
(ii) $\dot{p}(t) = -\partial H^*/\partial x = 0$;
(iii) $\dot{x}^*(t) = u^*(t)$, $x^*(0) = 1$, $x^*(3) = 3$.

From (ii) we get $p(t) = \bar{p}$ for some constant $\bar{p}$. Condition (i) implies that we must have $u^*(t) = 2$ if $\bar{p} > t - 4$ and $u^*(t) = 0$ if $\bar{p} < t - 4$. If $\bar{p} < t - 4$ for all t in $[0, 3]$, then $u^*(t) \equiv 0$, and from (iii) we have $x^*(t) \equiv 1$, contradicting $x^*(3) = 3$. In the same way we see that $\bar{p} > t - 4$ for all t in $[0, 3]$ is impossible. Hence we have to choose $u^*(t) = 2$ in some interval $[0, t^*]$ and $u^*(t) = 0$ in $(t^*, 3]$, with $t^* - 4 = \bar{p}$. Now from (iii) we have $x^*(t) = 2t + 1$ in $[0, t^*]$, and $x^*(t) = 2t^* + 1$ in $(t^*, 3]$. Since $x^*(3) = 2t^* + 1 = 3$, we see that $t^* = 1$, and then $p(t) = \bar{p} = t^* - 4 = -3$. It is clear that we have found the optimal solution, since all the conditions (i)–(iii) are satisfied.

(b) The Lagrangian is $\mathcal{L} = H + q(t + 1 - x) = (4 - t)u + pu + q(t + 1 - x)$. Here H is concave in (x, u) and $h(t, x) = t + 1 - x$ is quasiconcave, so the conditions (i) and (iii) in (a) in addition to

(ii)′ $q(t) \geq 0$, with $q(t) = 0$ if $t + 1 > x^*(t)$,
(iii)′ $\dot{p}(t) = -\partial \mathcal{L}^*/\partial x = q(t)$,
(iv) $p(3^-) - p(3) = -\beta$,
(v) $\beta \geq 0$, with $\beta = 0$ if $4 - x^*(3) > 0$,

are sufficient for $(x^*(t), u^*(t))$ to be optimal.

The objective function indicates that we should keep $u^*(t)$ as large as possible, especially at the beginning. But having $u^*(t)$ larger than 1 will cause $x^*(t)$ to violate the constraint $x \leq t + 1$, so we suggest $u^*(t) = 1$, and then $x^*(t) = t + 1$ in some interval $[0, t^*]$. Note that according to (i), $u^*(t) = 1$ can only maximize the Hamiltonian in $[0, t^*]$ provided $p(t) = t - 4$. From (iii)′ we further get $q(t) = 1$ in $[0, t^*]$. With $x^*(t) = t + 1$ we get $x^*(2) = 3$, and since $x^*(3) = 3$ and $\dot{x}^*(t) \geq 0$, we must have $t^* \leq 2$. In fact, we suggest $t^* = 2$ and then $u^*(t) = 0$ in $(2, 3]$. From $\dot{x}^*(t) \geq 0$, we get $x^*(t) \leq x^*(3) = 3$ and then $h(t, x^*(t)) = t + 1 - x^*(t) \geq t - 2$. It follows that $h(t, x^*(t)) > 0$ for t in $(2, 3]$. But by (ii)′ we have $q(t) = 0$ in $(2, 3]$. Then (iii)′ yields $p(t) = \bar{p}$ for some constant $\bar{p}$. Since $p(t)$ is continuous at $t = 2$, $p(2^-) = 2 - 4 = -2 = \bar{p}$. It remains to determine β. We see that $h(3, x^*(3)) = 3 + 1 - 3 = 1 > 0$, so from (v) we get $\beta = 0$, and (iv) gives $p(3^-) = p(3) = -2$.

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The optimal solution is spelled out the answer in the book. It is a useful exercise to check carefully that all the conditions (i), (ii)′, (iii)′, (iv), and (v) are now satisfied. Note in particular that for t in (2, 3] the expression $(p(t) - (t-4))u = (-2 - (t-4))u = (2-t)u$ is maximized by $u = u^*(t) = 0$, since $2 - t$ is negative in (2, 3].

11 Difference Equations

11.1

11.1.2 In parts (a)–(f) the solution is given by $x_t = a^t(x_0 - x^*) + x^*$, cf. formula (11.1.5). In (g)–(i) the solution is $x_t = x_0 + tb$.

(a) Since $0 < a < 1$, the power a^t decreases monotonically towards 0 as a limit as $t \to \infty$. Because $x_0 - x^* < 0$ it follows that x_t will increase monotonically towards the limit x^*.

(b) a^t alternates between negative and positive values and tends to 0. We get damped oscillations around the limit x^*.

(c) x_t increases monotonically and faster and faster towards ∞.

(d) Since $a < -1$, the powers a^t alternate between negative and positive values, while $|a^t|$ tends to ∞ as $t \to \infty$. Thus we get explosive oscillations about x^*.

(e) The solution is constant, $x_t = x^*$ for all t.

(f) Oscillations around x^* with constant amplitude.

(g) $x_t = x_0 + tb$ increases (linearly) towards ∞.

(h) The solution is monotonically (linearly) decreasing towards $-\infty$.

(i) $x_t = x_0$ for all t.

11.2

11.2.3 (a) Let the remaining debt on 1 January in year n be L_n. Then $L_0 = L$. Since the payment on the principal in year n is $L_{n-1} - L_n$ and the interest is rL_{n-1}, we have $L_{n-1} - L_n = \frac{1}{2}rL_{n-1}$, $n = 1, 2, \dots$. The solution is $L_n = (1 - \frac{1}{2}r)^n L$.

(b) $(1 - \frac{1}{2}r)^{10}L = \frac{1}{2}L$ implies that $r = 2 - 2 \cdot 2^{-1/10} \approx 0.133934$

(c) The payment in year n will be $L_{n-1} - L_n + rL_{n-1} = \frac{3}{2}rL_{n-1} = \frac{3}{2}r(1 - \frac{1}{2}r)^{n-1}L$. The loan will never be completely paid since $L_n > 0$ for all n (but it does tend to 0 in the limit as $n \to \infty$).

11.2.4 Let r_t be the interest rate in period t, a_t the repayment, and b_t the outstanding balance. Then $b_{t+1} = (1 + r_t)b_t - a_{t+1}$, where $b_0 = K$. We get

$$
\begin{aligned}
b_1 &= (1+r_0)b_0 - a_1 = (1+r_0)K - a_1 \\
b_2 &= (1+r_1)b_1 - a_2 = (1+r_1)(1+r_0)K - (1+r_1)a_1 - a_2 \\
b_3 &= (1+r_2)b_2 - a_3 = (1+r_2)(1+r_1)(1+r_0)K - (1+r_2)(1+r_1)a_1 - (1+r_2)a_2 - a_3 \\
&\quad \dots \quad \dots \quad \dots
\end{aligned}
$$

The pattern is clear, and it can be shown by induction that

$$
b_t = \prod_{s=0}^{t-1}(1+r_s)K - \sum_{s=1}^{t-1}\left[\prod_{k=s}^{t-1}(1+r_k)a_s\right] - a_t
$$

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It is reasonable to suppose that if the interest rate changes, then the repayments also change to ensure that $b_{t+1} < b_t$, i.e. $a_{t+1} > r_t b_t$. If the repayment in period $t+1$ is less than the interest accrued during period t, i.e. if $a_{t+1} < r_t b_t$, then the outstanding debt will increase, and if this situation continues, the loan will never be paid off.

11.3

11.3.1 (a) $x_{t+1} = A + B\,2^{t+1} = A + 2B\,2^t$ and $x_{t+2} = A + B\,2^{t+2} = A + 4B\,2^t$, so $x_{t+2} - 3x_{t+1} + 2x_t = A + 4B\,2^t - 3A - 6B\,2^t + 2A + 2B\,2^t = 0$ for all t.

(Section 11.4 shows how to find this solution, in the form $x_t = A1^t + B\,2^t$.)

(b) With $x_t = A3^t + B4^t$ we get $x_{t+1} = 3A3^t + 4B4^t$, $x_{t+2} = 9A3^t + 16B4^t$, and $x_{t+2} - 7x_{t+1} + 12x_t = 9A3^t + 16B4^t - 21A3^t - 28B4^t + 12A3^t + 12B4^t = 0$.

11.3.5 We shall prove that

$$u_t^{(1)} \text{ and } u_t^{(2)} \text{ are linearly dependent} \iff \begin{vmatrix} u_0^{(1)} & u_0^{(2)} \\ u_1^{(1)} & u_1^{(2)} \end{vmatrix} = 0$$

Proof of $\Rightarrow$: If the solutions $u_t^{(1)}$ and $u_t^{(2)}$ are linearly dependent, then there exist constants c_1 and c_2, not both equal to 0, such that $c_1 u_t^{(1)} + c_2 u_t^{(2)} = 0$ for all t. This holds, in particular, for $t = 0$ and $t = 1$, and so the columns of the determinant above are linearly dependent, and the determinant must be 0.

Proof of $\Leftarrow$: If the determinant is zero, the columns are linearly dependent, so there exist constants c_1 and c_2, not both 0, such that

$$c_1 u_t^{(1)} + c_2 u_t^{(1)} = 0 \tag{$*$}$$

for $t = 0$ and for $t = 1$. Now suppose that $(*)$ holds for $t = 0, 1, \dots, T-1$, where T is some integer greater than 1. Then

$$c_1 u_T^{(1)} + c_2 u_T^{(2)} = -a_t\bigl[c_1 u_{T-1}^{(1)} + c_2 u_{T-1}^{(2)}\bigr] - b_t\bigl[c_1 u_{T-2}^{(1)} + c_2 u_{T-2}^{(2)}\bigr] = 0$$

so $(*)$ holds for $t = T$ also. It follows by induction that $(*)$ holds for all $t \geq 0$. Hence, $u_t^{(1)}$ and $u_t^{(2)}$ are linearly dependent.

11.3.6 (a) From Problem 2 we can find the linearly independent solution $u_t^{(1)} = 1$ and $u_t^{(2)} = t$ of the homogeneous equation $x_{t+2} - 2x_{t+2} + x_t = 0$. Then $D_t = u_t^{(1)} u_{t+1}^{(2)} - u_{t+1}^{(1)} u_t^{(2)} = (t+1) - t = 1$ for all t, and we get

$$u_t^* = -u_t^{(1)} \sum_{k=1}^{t} c_{k-1} u_k^{(2)} + u_t^{(2)} \sum_{k=1}^{t} c_{k-1} u_k^{(1)} = -\sum_{k=1}^{t} k c_{k-1} + t \sum_{k=1}^{t} c_{k-1}$$

as a particular solution of $x_{t+2} - 2x_{t+1} + x_t = c_t$. The general solution is then $x_t = A + Bt + u_t^*$.

(b) With $c_t = t$, the particular solution u_t^* in part (a) becomes

$$u_t^* = -\sum_{k=1}^{t} k(k-1) + t \sum_{k=1}^{t} (k-1) = -\tfrac{1}{3}(t-1)t(t+1) + \tfrac{1}{2}t^2(t-1) = \tfrac{1}{6}t(t-1)(t-2)$$

The necessary summation formulas $\sum_{k=1}^{t} k(k-1) = \frac{1}{3}(t-1)t(t+1)$ and $\sum_{k=1}^{t}(k-1) = \frac{1}{2}t(t-1)$ are easily proved by induction. It is also easy to check that u_t^* really is a solution of the difference equation.

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11.4

11.4.2 (a) The characteristic equation $m^2 + 2m + 1 = (m + 1)^2 = 0$ has the double root $m = -1$, so the general solution of the associated homogeneous equation is $x_t^{\mathrm{H}} = (A + Bt)(-1)^t$. We find a particular solution of the nonhomogeneous equation by inserting $u_t^* = P2^t$. This yields $P = 1$, and so the general solution of the given equation is $x_t = (A + Bt)(-1)^t + 2^t$.

(b) By using the method of undetermined coefficients to determine the constants P, Q, and R in the particular solution $u_t^* = P5^t + Q\cos\frac{\pi}{2}t + R\sin\frac{\pi}{2}t$, we obtain $P = \frac{1}{4}$, $Q = \frac{3}{10}$, and $R = \frac{1}{10}$. So the general solution to the given equation is $x_t = A + B2^t + \frac{1}{4}5^t + \frac{3}{10}\cos\frac{\pi}{2}t + \frac{1}{10}\sin\frac{\pi}{2}t$.

11.4.4 Since $1 + a + b = 0$, we have $b = -1 - a$, so we are looking for a particular solution of the equation $x_{t+2} + ax_{t+1} - (a + 1)x_t = c$. A constant function will be a solution of the corresponding homogeneous function, so that will not work (unless $c = 0$). Let us try a function of the form $u_t^* = Dt$. We get

$$u_{t+2}^* + au_{t+1}^* - (a + 1)u_t^* = D(t + 2) + aD(t + 1) - (a + 1)Dt = D(a + 2)$$

Thus, $D = c/(a+2)$ can be used unless $a = -2$. If $a = -2$, the difference equation is $x_{t+2} - 2x_{t+1} + x_t = c$, and we look for a particular solution of the form $u_t^* = Dt^2$. In this case we get

$$u_{t+2}^* - 2u_{t+2}^* + u_t^* = D(t + 2)^2 - 2D(t + 1)^2 + Dt^2 = 2D$$

and the desired value of D is $D = c/2$.

11.4.6 If $b = \frac{1}{4}a^2$ and $x_t = u_t(-a/2)^t$, then the left-hand side of equation (11.4.1) becomes

$$\begin{aligned} x_{t+2} + ax_{t+1} + \tfrac{1}{4}a^2x_t &= u_{t+2}(-a/2)^{t+2} + au_{t+1}(-a/2)^{t+1} + \tfrac{1}{4}a^2u_t(-a/2)^t \\ &= \tfrac{1}{4}a^2(-a/2)^t(u_{t+2} - 2u_{t+1} + u_t) \end{aligned}$$

which is 0 if $u_{t+2} - 2u_{t+1} + u_t = 0$. The general solution of this equation is $u_t = A + Bt$, so $x_t = u_t(-a/2)^t = (A + Bt)(-a/2)^t$, which is the result claimed for case II in Theorem 11.4.1.

11.4.9 (a) It seems natural to try a function of the form $Y_t^* = C(1 + g)^t$. We get

$$Y_{t+2}^* - (b+k)Y_{t+1} + kY_t^* = C(1+g)^t[(1+g)^2 - (b+k)(1+g) + k] = C(1+g)^t[(1+g)^2 - b(1+g) - kg]$$

This shows that $Y_t^* = \dfrac{a(1 + g)^t}{(1 + g)^2 - (b + k)(1 + g) + k}$ (if the denominator of this fraction is nonzero).

(b) The equation $m^2 - (b + k)m + k = 0$ has two complex roots if and only if $(b + k)^2 - 4k < 0$.

(c) Part (III) of Theorem 11.4.1 shows that the growth factor of the oscillations is $r = \sqrt{k}$. The oscillations are damped if and only if $r < 1$, i.e. if and only if $k < 1$.

11.4.11 *Claim:* If $\frac{1}{4}a^2 \geq b$, then both roots of the equation $f(m) = m^2 + am + b = 0$ lie in the interval $(-1, 1)$ if and only if $|a| < 1 + b$ and $b < 1$.

Proof: Both roots belong to $(-1, 1)$ if and only if $f(-1) > 0$, $f(1) > 0$, $f'(-1) < 0$, and $f'(1) > 0$. (Draw a picture!) These four inequalities are equivalent to $1 - a + b > 0$, $1 + a + b > 0$, $-2 + a < 0$, and $a + 2 > 0$, which in turn are equivalent to $|a| < 1 + b$ and $|a| < 2$.

If $|a| < 2$, then $b \leq \frac{1}{4}a^2 < 1$. On the other hand, if $|a| < 1 + b$ and $b < 1$, then $|a| < 2$.

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11.5

11.5.3 The points (a_1, a_2) that satisfy the three inequalities in $(**)$ are the points that lie above each of the lines given by $a_2 = -1 - a_1$ and $a_2 = -1 + a_1$ and below the line $a_2 = 1$. These points are precisely the points lying in the interior of the triangle formed by those three lines, i.e. the triangle with corners at $(-2, 1)$, $(0, -1)$, and $(2, 1)$.

11.5.5 The characteristic equation is $m^2 + (\sigma\beta/\alpha - 2)m + (1 - \sigma\beta) = 0$. A necessary and sufficient condition for the roots of this equation to be complex (more precisely: not real) is

$$\left(\frac{\sigma\beta}{\alpha} - 2\right)^2 < 4(1 - \sigma\beta) \iff \frac{\sigma^2\beta^2}{\alpha^2} - \frac{4\sigma\beta}{\alpha} + 4 < 4 - 4\sigma\beta \iff \sigma^2\beta^2 < 4\alpha\sigma\beta - 4\alpha^2\sigma\beta$$
$$\iff \sigma\beta < 4\alpha - 4\alpha^2 = 4\alpha(1 - \alpha)$$

The difference equation is globally asymptotically stable if and only if the inequalities $(**)$ in Section 11.5 are satisfied when $a_1 = \sigma\beta/\alpha - 2$ and $a_2 = 1 - \sigma\beta$. This gives the conditions

$$1 + \left(\frac{\sigma\beta}{\alpha} - 2\right) + 1 - \sigma\beta > 0 \quad \text{and} \quad 1 - \left(\frac{\sigma\beta}{\alpha} - 2\right) + 1 - \sigma\beta > 0 \quad \text{and} \quad \sigma\beta > 0$$
$$\iff \quad \frac{\sigma\beta}{\alpha} - \sigma\beta > 0 \quad \text{and} \quad 4 > \frac{\sigma\beta}{\alpha} + \sigma\beta$$
$$\iff \quad \alpha < 1 \quad \text{and} \quad (1 + \alpha)\sigma\beta < 4\alpha$$

(Remember that α, β, and σ are all positive.)

11.6

11.6.1 (a) From the given equations we get $x_{t+2} = 2y_{t+1} = x_t$, and the initial conditions give $x_0 = 1$, $x_1 = 2y_0 = 2$. The characteristic equation of $x_{t+2} - x_t = 0$ is $m^2 - 1 = 0$, which has the roots $m_1 = 1$, $m_2 = -1$.

The general solution of $x_{t+2} - x_t = 0$ is $x_t = A + B(-1)^t$, and the initial conditions imply $A + B = 1$, $A - B = 2$, so $A = \frac{3}{2}$ and $B = -\frac{1}{2}$. Thus, $x_t = \frac{3}{2} - \frac{1}{2}(-1)^t$. This, in turn, gives $y_t = \frac{1}{2}x_{t+1} = \frac{3}{4} + \frac{1}{4}(-1)^t$.

(b) We first eliminate z. The first equation yields $z_t = -x_{t+1} - y_t + 1$. Using this in the second and third equations, we get

$$\text{(i)} \;\; y_{t+1} = -x_t + x_{t+1} + y_t - 1 + t \quad \text{and} \quad \text{(ii)} \;\; -x_{t+2} - y_{t+1} + 1 = -x_t - y_t + 2t$$

Equation (ii) implies (iii) $y_{t+1} = -x_{t+2} + 1 + x_t + y_t - 2t$, and then (i) and (iii) imply

$$-x_t + x_{t+1} + y_t - 1 + t = -x_{t+2} + 1 + x_t + y_t - 2t, \quad \text{and so} \quad \text{(iv)} \;\; x_{t+2} + x_{t+1} - 2x_t = 2 - 3t$$

By a stroke of good luck y does not appear in (iv). The characteristic equation of (iv) is $m^2 + m - 2$ with the roots $m_1 = 1$ and $m_2 = -2$, and so the homogeneous equation corresponding to (iv) has the general solution $x_t^{\mathrm{H}} = A + B(-2)^t$. For a particular solution u_t^* of (iv) itself we try with a quadratic polynomial $u^* = Dt + Et^2$. We get $u_t^* = \frac{3}{2}t - \frac{1}{2}t^2$, so the general solution of (iv) is $x_t = x_t^{\mathrm{H}} + u_t^* = A + B(-2)^t + \frac{3}{2}t - \frac{1}{2}t^2$.

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The initial conditions yield $x_0 = 0$ and $x_1 = -y_0 - z_0 + 1 = 0$, so we must have $A + B = 0$, $A - 2B + \frac{3}{2} - \frac{1}{2} = 0$, which implies, $A = -\frac{1}{3}$, $B = \frac{1}{3}$. Hence,

$$x_t = -\tfrac{1}{3} + \tfrac{1}{3}(-2)^t + \tfrac{3}{2}t - \tfrac{1}{2}t^2$$

From equation (i) we get $y_{t+1} - y_t = x_{t+1} - x_t + t - 1 = -(-2)^t$ with the general solution $y_t = A + \frac{1}{3}(-2)^t$. The initial condition $y_0 = 0$ yields $A = -\frac{1}{3}$, so

$$y_t = -\tfrac{1}{3} + \tfrac{1}{3}(-2)^t$$

Finally,

$$z_t = -x_{t+1} - y_t + 1 = \tfrac{2}{3} + \tfrac{1}{3}(-2)^t - \tfrac{1}{2}t + \tfrac{1}{2}t^2$$

11.7

11.7.2 (a) Let $f(x) = e^x - 3$. Then $f(x) - x$ is convex, and it is easy to see from the intermediate value theorem that the equation $f(x) = x$ has two solutions, one in $(-\infty, 0)$ and one in $(0, \infty)$. Since $|f'(x)| = e^x < 1$ for all $x < 0$, the negative solution is a stable equilibrium of the difference equation $x_{t+1} = f(x_t)$. The solution of the difference equation starting at $x_0 = -1$ gives the values (rounded to 5 decimal places) $x_1 = -2.63212$, $x_2 = -2.92807$, $x_3 = -2.94650$, $x_4 = -2.94748$, $x_5 = -2.94753$, $x_6 = -2.94753, \ldots$, converging to the equilibrium value $x^* \approx -2.94753$.

(b) See the answer in the book.

12 Discrete Time Optimization

12.1

12.1.1 (a) To solve the problem by dynamical programming we first find

$$J_2(x) = \max_u (1 - (x^2 + 2u^2)) = 1 - x^2, \qquad u_2^*(x) = 0$$

The fundamental equation (Theorem 12.1.1) then gives

$$\begin{aligned} J_1(x) &= \max_u (1 - (x^2 + 2u^2) + J_2(x - u)) \\ &= \max_u (1 - x^2 - 2u^2 + 1 - (x - u)^2) \\ &= \max_u (\underbrace{2 - 2x^2 + 2xu - 3u^2}_{g(u)}) \end{aligned}$$

Let $g(u)$ be the expression in the last parenthesis. Then $g'(u) = 2x - 6u$, so g attains its maximum for $u = u_1^*(x) = x/3$. That gives $J_1(x) = g(x/3) = 2 - \frac{5}{3}x^2$. We continue with

$$\begin{aligned} J_0(x) &= \max_u (1 - (x^2 + 2u^2) + J_1(x - u)) \\ &= \cdots = \max_u (\underbrace{3 - \frac{8}{3}x^2 + \frac{10}{3}xu - \frac{11}{3}u^2}_{h(u)}) \end{aligned}$$

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Here $h'(u) = 10x/3 - 22u/3$, which implies that $u_0^*(x) = 5x/11$, and therefore $J_0(x) = h(5x/11) = 3 - 21x^2/11$.

Thus the desired maximum value is $J_0(x_0) = J_0(5) = -492/11$ and, further,

$$u_0^* = u_0(5) = 25/11, \qquad x_1^* = x_0 - u_0^* = 30/11$$
$$u_1^* = u_1^*(x_1^*) = x_1^*/3 = 10/11, \qquad x_2^* = x_1^* - u_1^* = 20/11$$

(b) We have $x_1 = x_0 - u_0 = 5 - u_0$ and $x_2 = x_1 - u_1 = 5 - u_0 - u_1$. This gives

$$\begin{aligned} S(u_0, u_1, u_2) &= 1 - (x_0^2 + 2u_0^2) + 1 - (x_1^2 + 2u_1^2) + 1 - (x_2^2 + 2u_2^2) \\ &= 3 - 5^2 - 2u_0^2 - (5 - u_0)^2 - 2u_1^2 - (5 - u_0 - u_1)^2 - 2u_2^2 \\ &= \cdots = -72 + 20u_0 + 10u_1 - 4u_0^2 - 2u_0u_1 - 3u_1^2 - 2u_2^2 \end{aligned}$$

It is clear from the second expression for S that S is a concave function. The first-order partial derivatives of S are

$$\begin{aligned} \partial S/\partial u_0 &= 20 - 8u_0 - 2u_1 \\ \partial S/\partial u_1 &= 10 - 2u_0 - 6u_1 \\ \partial S/\partial u_0 &= -4u_2 \end{aligned}$$

and it is easily seen that the only stationary point is

$$(u_0, u_1, u_2) = (25/11,\ 10/11,\ 0)$$

Since S is concave, this is a global maximum point for S. The maximum value is $S_{\max} = -492/11$, which fortunately agrees with the result from part (a).

12.1.4 (a) $J_T(x) = 3x^2$ with $u_T^*(x) = 0$, $J_{T-1}(x) = 5x^2$ with $u_{T-1}^*(x) = 1$, $J_{T-2}(x) = 7x^2$ with $u_{T-2}^*(x) = 1$.

(b) We claim that $J_{T-n}(x) = (2n+3)x^2$ with $u_T^*(x) = 0$ and $u_{T-n}^*(x) = 1$ for $n = 1, \ldots, T$. The formula is valid for $n = 1$. Suppose it is valid for $n = k$. Then $J_{T-(k+1)}(x) = \max_{u\in[0,1]}[(3-u)x^2 + J_{T-k}(ux)] = \max_{u\in[0,1]}[(3-u)x^2 + (2k+3)(ux)^2] = x^2 \max_{u\in[0,1]}[3 - u + (2k+3)u^2]$. The function $g(u) = 3 - u + (2k+3)u^2$ is convex in u and has its maximum at $u = 1$, and then $J_{T-(k+1)}(x) = x^2(5+2k)$, which is the proposed formula for $n = k+1$, so the formula follows by induction.

12.1.6 (a) $J_T(x) = \max_{u\in\mathbb{R}}(x - u^2) = x$ for $u_T^*(x) = 0$. $J_s(x) = \max_{u\in\mathbb{R}}[x - u^2 + J_{s+1}(2(x+u))]$ for $s = 0, 1, \ldots, T-1$. In particular, $J_{T-1}(x) = \max_{u\in\mathbb{R}}[x - u^2 + J_T(2(x+u))] = \max_{u\in\mathbb{R}}[x - u^2 + 2(x+u)] = 3x + 1$ for $u_{T-1}^*(x) = 1$.

(b) The formula is valid for $n = 1$. Suppose it is valid for $n = k$. Then $J_{T-(k+1)}(x) = \max_{u\in\mathbb{R}}[x - u^2 + (2^{k+1} - 1)(2x + 2u) + \sum_{j=0}^{k}(2^j - 1)^2]$. We see that the maximizer is $u = u_{T-(k+1)}^*(x) = 2^{k+1} - 1$, and then $J_{T-(k+1)}(x) = x - (2^{k+1} - 1)^2 + 2(2^{k+1} - 1)x + 2(2^{k+1} - 1)^2 + \sum_{j=0}^{k}(2^j - 1)^2 = (1 + 2^{k+2} - 2)x + (2^{k+1} - 1)^2 + \sum_{j=0}^{k}(2^j - 1)^2 = (2^{(k+1)+1} - 1)x + \sum_{j=0}^{k+1}(2^j - 1)^2$. This is the given formula for $n = k+1$, so the formula follows by induction. Since $u_{T-(k+1)}^*(x) = 2^{k+1} - 1$, we get $u_t^*(x) = 2^{T-t} - 1$ for $t = 0, 1, \ldots, T$, and $V = J_0(x_0) = J_0(0) = \sum_{j=0}^{T}(2^j - 1)^2$.

12.1.7 (a) It is immediately clear that $J_T(x) = -\alpha e^{-\gamma x_T}$. The result in part (b) shows immediately that $J_{T-1}(x) = -2\sqrt{\alpha}e^{-\gamma x}$ and $J_{T-2}(x) = -2\sqrt{2\sqrt{\alpha}}e^{-\gamma x} = -2^{3/2}\alpha^{1/4}e^{-\gamma x}$.

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(b) We know from part (a) that the formula $J_t(x) = -\alpha_t = -\alpha_t e^{-\gamma x}$ holds for $t = T$ with $\alpha_t = \alpha$. Suppose that it holds for a positive integer $t \le T$. The fundamental equation then gives $J_{t-1}(x) = \max_{u\in\mathbb{R}} \varphi(u)$, where

$$\varphi(u) = -e^{-\gamma u} + J_t(2x-u) = -e^{-\gamma u} - \alpha_t e^{-\gamma(2x-u)}$$

The function φ is concave and

$$\begin{aligned}\varphi'(u) = 0 \iff \gamma e^{-\gamma u} - \alpha_t \gamma e^{-\gamma(2x-u)} = 0 &\iff -\gamma u = \ln\alpha_t - 2\gamma x + \gamma u \\ &\iff \gamma u = \gamma x - \ln\sqrt{\alpha_t}\end{aligned}$$

This shows that φ has a unique stationary point $u^* = x - (\ln\sqrt{\alpha_t}\,)/\gamma$, which is a maximum point for φ. It follows that

$$J_{t-1}(x) = \varphi(u^*) = -e^{-\gamma u^*} - \alpha_t e^{-2\gamma x + \gamma u^*} = -e^{\ln\sqrt{\alpha_t} - \gamma x} - \alpha_t e^{-\gamma x - \ln\sqrt{\alpha_t}} = -\alpha_{t-1} e^{-\gamma x}$$

where $\alpha_{t-1} = \sqrt{\alpha_t} + (\alpha_t/\sqrt{\alpha_t}\,) = 2\sqrt{\alpha_t}$. It follows by induction that the formula $J_t(x) = -\alpha_t e^{-\gamma x}$ holds for $t = T, T-1, \dots, 1, 0$, with α_t determined by the difference equation above and $\alpha_T = \alpha$.

12.3

12.3.1 The equation $\alpha = 2\sqrt{\alpha\beta} + \frac{1}{2}$ can be written as $(\sqrt{\alpha}\,)^2 - 2\sqrt{\beta}\sqrt{\alpha} - \frac{1}{2} = 0$. This is a quadratic equation for $\sqrt{\alpha}$ with the solution $\sqrt{\alpha} = \sqrt{\beta} + \sqrt{\beta + 1/2}$. (We cannot have $\sqrt{\alpha} = \sqrt{\beta} - \sqrt{\beta + 1/2}$, because $\sqrt{\alpha}$ cannot be negative.) Hence, $\alpha = (\sqrt{\beta} + \sqrt{\beta + 1/2}\,)^2$.

To show optimality, we can use Case B in Note 12.7.3. We first solve the corresponding finite horizon problem

$$\sup_{u_t} \sum_{t=0}^{T} \beta^t(-e^{-u_t} - \tfrac{1}{2}e^{-x_t}), \quad x_{t+1} = 2x_t - u_t, \;\; t = 0, 1, \dots, T-1, \quad x_0 \text{ given}$$

With the optimal value function $J(t, x, T) = \sup \sum_{s=t}^{T} \beta^{s-t}(-e^{-u_s} - \frac{1}{2}e^{-x_s})$ we get $J(t, x, T) = -\alpha_t e^{-x}$, with $\alpha_T = \frac{1}{2}$ and $\alpha_t = 2\sqrt{\beta\alpha_t} + \frac{1}{2}$ for $t < T$. For a fixed T, $\alpha_t \to \alpha$ as $t \to -\infty$. Hence, $J(0, x, T) \to -\alpha e^{-x}$ as $T \to \infty$.

12.3.2 (a) The Bellman equation is $J(x) = \max_{u\in\mathbb{R}}\bigl[-\frac{2}{3}x^2 - u^2 + \beta J(x+u)\bigr]$. If $J(x) = -\alpha x^2$ is a solution, then

$$-\alpha x^2 = \max_{u\in\mathbb{R}}\bigl[-\tfrac{2}{3}x^2 - u^2 + \beta J(x+u)\bigr] = \max_{u\in\mathbb{R}} \varphi(u) \tag{$*$}$$

where $\varphi(u) = -\frac{2}{3}x^2 - u^2 - \beta\alpha(x+u)^2$. The function φ is strictly concave, and has a unique maximum point given by $\varphi'(u) = -2u - 2\beta\alpha(x+u) = 0$. Hence, $u^*(x) = -\beta\alpha x/(1+\beta\alpha)$ and $x + u^*(x) = x/(1+\beta\alpha)$. Equation $(*)$ now gives

$$\begin{aligned}-\alpha x^2 = \varphi(u^*(x)) &= -\frac{2}{3}x^2 - \frac{\beta^2\alpha^2}{(1+\beta\alpha)^2}x^2 - \frac{\beta\alpha}{(1+\beta\alpha)^2}x^2 = -\frac{2}{3}x^2 - \frac{\beta^2\alpha^2 + \beta\alpha}{(1+\beta\alpha)^2}x^2 \\ \iff \alpha &= \frac{2}{3} + \frac{\beta\alpha(\beta\alpha+1)}{(1+\beta\alpha)^2} = \frac{2}{3} + \frac{\beta\alpha}{1+\beta\alpha} \\ \iff 3\alpha(1+\beta\alpha) &= 2(1+\beta\alpha) + 3\beta\alpha \iff 3\beta\alpha^2 + (3-5\beta)\alpha - 2 = 0\end{aligned}$$

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The last equation has a exactly one positive solution, $\alpha = \dfrac{5\beta - 3 + \sqrt{(5\beta-3)^2 + 24\beta}}{6\beta}$, and the optimal control is

$$u^*(x) = -\frac{\beta\alpha}{1+\beta\alpha}\,x = -\frac{5\beta - 3 + \sqrt{(5\beta-3)^2+24\beta}}{5\beta + 3 + \sqrt{(5\beta-3)^2+24\beta}}\,x = \cdots = -\frac{\beta - 3 + \sqrt{(5\beta-3)^2+24\beta}}{6\beta}\,x$$

(b) It is clear that the value function $J_0(x)$ is finite: for any x_0, let $u_0 = -x_0$ and $u_t = 0$ for all $t > 0$. Then $\sum_{t=0}^{\infty} \beta^t(-\frac{2}{3}x_t^2 - u_t^2) = -\frac{5}{3}x_0^2$ is finite. Also, for any sequence of controls $u_0, u_1, \dots$, the sum is bounded above by 0. Hence, the value function exists, if not in the "max sense", then at least in the "sup sense". It is also clear that $J_0(x) \le 0$ for all x.

Further, let $V_0(x, \pi)$ be the sum that results from $x = x_0$ and the control sequence $\pi = (u_0, u_1, \dots)$. Then for any number λ we get $V_0(\lambda x, \lambda\pi) = \lambda^2 V_0(x, \pi)$, and it follows that $J_0(\lambda x) = \lambda^2 J_0(x)$—that is, $J = J_0$ is homogeneous of degree 2, so we really do have $J(x) = -\alpha x^2$ for a suitable $\alpha \ge 0$. It is also clear that $\alpha \ne 0$, and from the arguments above it follows that $\alpha < \frac{5}{3}$.

Now let x be fixed, and consider the problem of finding a u that maximizes

$$\varphi(u) = -\tfrac{2}{3}x^2 - u^2 + J(x+u)$$

If $|x+u| > |x|$, then $J(x+u) < J(x)$ and

$$\varphi(u) = -\tfrac{2}{3}x^2 - u^2 + \beta J(x+u) < -\tfrac{2}{3}x^2 - u^2 + \beta J(x) < -\tfrac{2}{3}x^2 + \beta J(x) = \varphi(0)$$

Hence, such a u cannot be optimal. Also, if $|u| > |x|$, then $J(x+u) < 0$ and

$$\varphi(u) = -\tfrac{2}{3}x^2 - u^2 + \beta J(x+u) < -\tfrac{2}{3}x^2 - u^2 < -\tfrac{5}{3}x^2 = \varphi(-x)$$

so this u cannot be optimal either. It follows that an optimal u must be such that $|x+u| \le |x|$ and $|u| \le |x|$. Then Note 12.3.2 applies with $\mathcal{X}(x_0) = [-|x_0|, |x_0|]$.

12.4

12.4.2 (a) $I = \sum_{t=0}^{T}(u_t^2 - 2x_t^2) = \sum_{t=0}^{T-1} u_t^2 + u_T^2 - 2x_0^2 - 2\sum_{t=1}^{T} x_t^2 = u_T^2 + \sum_{t=0}^{T-1} u_t^2 - 2\sum_{t=0}^{T-1} u_t^2 = u_T^2 - \sum_{t=0}^{T-1} u_t^2$. (Remember, $x_0 = 0$.) Hence, I is maximized when $u_0^* = u_1^* = \cdots = u_{T-1}^* = 0$ and $u_T^* = \pm 1$.

(b) (The reference in the problem should be to Theorem 12.4.1, not 12.4.2.) The Hamiltonian is

$$H(t, x, u, p) = \begin{cases} u^2 - 2x^2 + pu & \text{for } t < T \\ u^2 - 2x^2 & \text{for } t = T \end{cases}$$

and

$$H'_u(t, x, u, p) = \begin{cases} 2u + p & \text{for } t < T \\ 2u & \text{for } t = T \end{cases}, \qquad H'_x(t, x, u, p) = -4x \quad \text{for all } t$$

Since $x_0^* = x_1^* = \cdots = x_T^* = 0$ and $u_0^* = u_1^* = \cdots = u_{T-1}^* = 0$, the difference equation (4) in Theorem 12.4.1 implies $p_t = 0$ for $t = 0, 1, \dots, T-1$, and we already know that $p_T = 0$. It follows that $H(t, x_t^*, u, p_t) = u^2 - 2(x_t^*)^2 = u^2$. Hence, for $t < T$, $u = u_t^* = 0$ is not a maximum point of $H(t, x_t^*, u, p_t)$ for u in $[-1, 1]$, but actually a minimum point.

12.5

12.5.1 (c) $H(t, x, u, p) = 1 + x - y - 2u^2 - v^2 + p^1(x - u) + p^2(y + v)$ for $t = 0, 1$, $H(t, x, u, p) = 1 + x - y - 2u^2 - v^2$ for $t = 2$. Condition (3) yields for $t = 0, 1$: $-4u_t - p_t^1 = 0$ and $-2v_t + p_t^2 = 0$.

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For $t = 2$ it yields $-4u_2 = 0$ and $-2v_2 = 0$. Hence, $u_0 = -\frac{1}{4}p_0^1$, $u_1 = -\frac{1}{4}p_1^1$, $u_2 = 0$, $v_0 = \frac{1}{2}p_0^2$, $v_1 = \frac{1}{2}p_1^2$, $v_2 = 0$. From (4) and (6)(c′), $p_0^1 = 1 + p_1^1$, $p_2^1 = 0$, $p_0^2 = -1 + p_1^2$, $p_2^2 = 0$. Moreover, from (5), $p_1^1 = 1 + p_2^1$, $p_1^2 = -1 + p_2^2$. Finally, $x_1 = x_0 - u_0 = 5 - u_0$, $x_2 = x_1 - u_1$, $y_1 = y_0 + v_0$, $y_2 = y_1 + v_1$. From these equations we find the same solution as before.

12.6

12.6.6 There are two misprints in the objective function: the expression inside the square brackets should be $\sum_{t=0}^{T-1} u_t^{1/2} + aX_T^{1/2}$. There is also a misprint in the answer in the book.

The optimality equation (12.6.5) boils down to

$$J_T(x) = ax^{1/2} \qquad \text{and} \qquad J_t(x) = \max_u \big[u^{1/2} + \tfrac{1}{2}J_{t+1}(0) + \tfrac{1}{2}J_{t+1}(x-u)\big] \quad \text{for} \quad t < T$$

With $J_t(x) = 2a_t x^{1/2}$ this gives $a_T = a/2$ and

$$2a_t x^{1/2} = \max_u \varphi(u), \quad \text{where} \quad \varphi(u) = u^{1/2} + a_{t+1}(x-u)^{1/2}$$

for $t < T$. The function φ is concave and

$$\varphi'(u) = 0 \iff \frac{1}{2u^{1/2}} = \frac{a_{t+1}}{2(x-u)^{1/2}} \iff u = u_t(x) = \frac{x}{1 + a_{t+1}^2}$$

Hence, $\max_u \varphi(u) = \varphi\big(x/(1 + a_{t+1}^2)\big) = \cdots = (1 + a_{t+1}^2)^{1/2} x^{1/2}$, and so $a_t = \frac{1}{2}(1 + a_{t+1}^2)^{1/2}$. This implies $1 + a_{t+1}^2 = 4a_t^2$, and therefore $u_t(x) = x/4a_t^2$.

12.6.8 The first printing of the book contains a couple of embarrassing misprints in connection with this problem and with the stochastic Euler equation. Formula (12.6.10) on page 454 should be

$$E\big[F_2'(t, x_t, x_{t+1}(x_t, V_t), V_t) \mid v_{t-1}\big] + F_3'(t-1, x_{t-1}, x_t, v_{t-1}) = 0 \tag{$*$}$$

and the objective function in the current problem should be

$$\max\ E\Big[\sum_{t=0}^{2}\big[1 - (V_t + X_{t+1} - X_t)^2\big] + (1 + V_2 + X_3)\Big]$$

Now define F by

$$F(2, x_2, x_3, v_2) = 1 - (v_2 + x_3 - x_2)^2 + 1 + v_2 + x_3$$
$$F(t, x_t, x_{t+1}, v_t) = 1 - (v_t + x_{t+1} - x_t)^2, \qquad t = 0, 1$$

For $t = 1, 2$, equation $(*)$ becomes

$$E\big[2(V_t + x_{t+1}(x_t, V_t) - x_t) \mid v_{t-1}\big] - 2(v_{t-1} + x_t - x_{t-1}) = 0$$

i.e.

$$1 + 2E[x_{t+1}(x_t, V_t)] - 2x_t = 2v_{t-1} + 2x_t - 2x_{t-1} \tag{$**$}$$

Equation (12.6.9) becomes

$$F_3'(2, x_2, x_3, v_2) = -2(v_2 + x_3 - x_2) + 1 = 0$$

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which gives x_3 as a function of x_2 and v_2:

$$x_3 = x_3(x_2, v_2) = x_2 - v_2 + \tfrac{1}{2}$$

Now use (∗∗) for $t = 2$:

$$1 + 2E[x_2 - V_2 + \tfrac{1}{2}] - 2x_2 = 2v_1 + 2x_2 - 2x_1 \iff 1 + 2(x_2 - \tfrac{1}{2} + \tfrac{1}{2}) - 2x_2 = 2v_1 + 2x_2 - 2x_1$$
$$\iff x_2 = x_2(x_1, v_1) = x_1 - v_1 + \tfrac{1}{2}$$

Then for $t = 1$:

$$1 + 2E[x_1 - V_1 + \tfrac{1}{2}] - 2x_1 = 2v_0 + 2x_1 - 2x_0 \iff 1 + 2(x_1 - \tfrac{1}{2} + \tfrac{1}{2}) - 2x_1 = 2v_0 + 2x_1 - 2x_0$$
$$\iff x_1 = x_0 - v_0 + \tfrac{1}{2}$$

Since $x_0 = 0$ is given, the final answer is

$$x_1 = \tfrac{1}{2} - v_0, \quad x_2 = 1 - v_0 - v_1, \quad x_3 = \tfrac{3}{2} - v_0 - v_1 - v_2$$

12.7

12.7.1 (a) $J(x) = ax^2 + b$, $a = -[1 - 2\beta - \sqrt{1 + 4\beta^2}\,]/2\beta$, $b = a\beta d/(1-\beta)$. (With $J(x) = ax^2 + b$, the Bellman equation is

$$ax^2 + b = \max_u \bigl\{-u^2 - x^2 + \beta E[a(x + u + V)^2 + b]\bigr\} = \max_u \bigl\{-u^2 - x^2 + \beta a(x+u)^2 + \beta a d + \beta b\bigr\}$$

Maximizing this concave function yields $u = \beta a x/(1 - \beta a)$. Thus, $ax^2 + b = -\beta^2 a^2 x^2/(1-\beta a)^2 - x^2 + \beta a x^2/(1-\beta a)^2 + \beta a d + \beta b = x^2(2\beta a - 1)/(1 - \beta a) + \beta a d + \beta b$ for all x. Hence, $a = (2\beta a - 1)/(1 - \beta a)$ and $b = \beta a d + \beta b$. This quadratic equation gives $a = [1 - 2\beta - \sqrt{1 + 4\beta^2}\,]/2\beta$, and then $b = a\beta d/(1-\beta)$. We have to choose the negative solution for a, because $J(x) = ax^2 + b = a[x^2 + \beta d/(1-\beta)]$ must be negative.)

(b) $J(t, x) = \beta^t(a_t x^2 + b_t)$, $a_{t-1} = -1 - \beta^2 a_t^2/(1 - \beta a_t)^2 + \beta a_t/(1-\beta a_t)^2 = -1 + \beta a_t/(1 - \beta a_t)$, $a_T = -1$, $b_{t-1} = \beta b_t + \beta a_t d$, $b_T = 0$. To find $\lim_{T\to\infty} J(0, x_0, T)$ we need to find $\lim_{T\to\infty} a_0$ and $\lim_{T\to\infty} b_0$ (for any t, a_t and b_t depend on T), write in particular $a_0 = a_0^T$, $b_0 = b_0^T$. Finding these limits is the same as finding the limits $\lim_{t\to-\infty} a_t$, $\lim_{t\to-\infty} b_t$ when T is fixed. The function $\varphi(x) = -1 + \beta x/(1 - \beta x)$ is increasing (calculate its derivative), and, since $a_{T-1} < a_T$ and this continues backwards, we get $a_{t-1} < a_t$ for all t. Letting $t \to -\infty$ in the difference equation for a_t, we find that $a = \lim_{t\to-\infty} a_t$ satisfies $a = -1 + \beta a/(1 - \beta a) = (2\beta a - 1)/(1 - \beta a)$ (so $a > -\infty$), in fact a has the same value as in part (a). In a similar way, b_t decreases when t decreases, and taking limits in the equation for b_{t-1}, we find that $b = \lim_{t\to-\infty} b_t$ satisfies $b = \beta b + \beta a d$, i.e. b is also as in part (a). Then, evidently, $J(0, x, T) = a_0^T x^2 + b_0^T \to ax^2 + b = J(x)$ as $T \to \infty$.

12.7.2 The optimal control is $u_t(x) = x/(1 + \alpha a)$, and the value function is $J(x) = a \ln x + b$, where $a = 2/(1-\alpha)$, $b = [\alpha d + \alpha a \ln(\alpha a) - (1 + \alpha a)\ln(1 + \alpha a)](1-\alpha)^{-1}$, and $d = E[\ln V]$.

We can show optimality at least in a restricted problem: Assume $V \in [0, \delta]$ for some (perhaps large) δ, and restrict u to belong to $[\varepsilon, x_t]$ for some small positive ε. Note that $X_t \geq x_0$. Then $|f| = |(x - u)V| \leq \delta x$. Choose $b > 0$ so small that $\alpha\delta^b < 1$. For $x \geq x_0$, $g = \ln u + \ln X \in [\ln\varepsilon + \ln x_0, 2\ln x] \subseteq [\ln\varepsilon + \ln x_0, \ln a + x^b]$, where a is chosen so large that $\ln x/x^b \leq 1$ when $x \geq a$ (by l'Hôpital's rule $\lim_{x\to\infty} \ln x/x^b = 0$). Now apply Note 12.7.2.

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13 Topology and Separation

13.1

13.1.9 (a) If $\mathbf{x} \in \text{int}(S)$, then there is an open ball B around $\mathbf{x}$ such that $B \subseteq S$. But then $B \subseteq T$ as well, so $\mathbf{x}$ is an interior point of T. If $\mathbf{y} \in \text{cl}(S)$, then every open ball around $\mathbf{y}$ has a nonempty intersection with S. Obviously any such ball also meets T, and so $\mathbf{y}$ belongs to the closure of T.

(b) Let $\mathbf{y}$ be a point in the closure of $\text{cl}(S)$. We want to show that $\mathbf{y} \in \text{cl}(S)$. In order to show this, it is sufficient to show that every open ball around $\mathbf{y}$ intersects S. Thus, let B be an open ball around $\mathbf{y}$. Since $\mathbf{y}$ is in the closure of $\text{cl}(S)$, the intersection $B \cap \text{cl}(S)$ is nonempty. Let $\mathbf{z}$ be any point in this intersection. Since $\mathbf{z} \in B$ and B is open, there is an open ball B' around $\mathbf{z}$ such that $B' \subseteq B$. And since $\mathbf{z} \in \text{cl}(S)$, there is at least one point $\mathbf{w}$ in $B' \cap S$. Then $\mathbf{w} \in B \cap S$, and so $B \cap S \neq \emptyset$. Hence, $\text{cl}(S)$ is closed.

13.1.15 (a) False. Since $S \subseteq \overline{S}$, it is clear that $\text{int}(S) \subseteq \text{int}(\overline{S})$. But we do not always have equality, as we can see from the following example: Let $S = \{\mathbf{x} \in \mathbb{R}^n : 0 < \|\mathbf{x}\| < 1\}$. This set (a "punctured" open ball) is obviously open, so $\text{int}(S) = S$. But its closure is $\overline{S} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ whose interior, $\text{int}(\overline{S}) = \{\|\mathbf{x}\| \in \mathbb{R}^n : \|\mathbf{x}\| < 1\}$, also contains the centre of the ball, so $\text{int}(\overline{S}) \neq \text{int}(S)$. (Draw a picture for the case $n = 2$!)

(b) True. Every set is contained in its closure, so $S \subseteq \overline{S}$ and $T \subseteq \overline{T}$. Therefore, $S \cup T \subseteq \overline{S} \cup \overline{T}$. By Theorem 13.1.2(c), the set $\overline{S} \cup \overline{T}$ is closed. Therefore, $\overline{S \cup T} \subseteq \overline{S} \cup \overline{T}$ (cf. Problem 10(b)). On the other hand, $S \subseteq S \cup T$, so by Problem 9(a), $\overline{S} \subseteq \overline{S \cup T}$. Similarly, $\overline{T} \subseteq \overline{S \cup T}$, and so $\overline{S} \cup \overline{T} \subseteq \overline{S \cup T}$. Since each of the sets $\overline{S} \cup \overline{T}$ and $\overline{S \cup T}$ is a subset of the other, they must be equal.

(c) False. Consider, for example, the punctured open ball S in part (a). The boundary, ∂S, of that set consists of the origin and all the points on the sphere $\|\mathbf{x}\| = 1$, so ∂S is not a subset of S. In fact, for any set T, we have $\partial T \subseteq T$ if and only if T is closed.

(d) True. Let $\mathbf{x} \in S \cap \overline{T}$. We shall show that $\mathbf{x} \in \overline{S \cap T}$. Let U be an arbitrary open ball centred at $\mathbf{x}$. It is enough to show that U intersects $S \cap T$. Since $\mathbf{x} \in S$ and S is open, there exists an open ball V centred at $\mathbf{x}$ such that $V \subseteq S$. Then $W = U \cap V$ is also an open ball centred at $\mathbf{x}$ and $W \subseteq S$. (In fact, W is the smaller of the two balls U and V.) Now, $W \cap T \neq \emptyset$ since $x \in \overline{T}$. Moreover, $W \cap T = U \cap V \cap T \subseteq U \cap S \cap T$, so it follows that $U \cap (S \cap T)$ is indeed nonempty.

13.2

13.2.5 Suppose for a contradiction that the sequence $\{\mathbf{x}_k\}$ does not converge to $\mathbf{x}^0$. Then there exists an open ball $B = B_r(\mathbf{x}^0)$ around $\mathbf{x}^0$ such that $\mathbf{x}_k \notin B_r(\mathbf{x}^0)$ for infinitely many k. These $\mathbf{x}_k$ form a subsequence $\{\mathbf{x}_{k_j}\}$ of the original sequence, and they all belong to the set $A = X \setminus B = X \cap (\mathbb{R}^n \setminus B)$. Since X is closed and B is open, A is closed. Because A is contained in the bounded set X, A is also bounded. Hence A is compact. Therefore $\{\mathbf{x}_{k_j}\}$ has a convergent subsequence, converging to a point $\mathbf{y}$ in A. But this convergent subsequence is also a subsequence of the original sequence, and so it should converge to $\mathbf{x}^0$. But that is impossible because $\mathbf{y} \neq \mathbf{x}^0$. This contradiction shows that $\{\mathbf{x}_k\}$ must converge to $\mathbf{x}^0$ after all.

13.2.6 There is an obvious way to identify $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$: we let the point $(\mathbf{x}, \mathbf{y}) = ((x_1, \ldots, x_m), (y_1, \ldots, y_n))$ in $\mathbb{R}^m \times \mathbb{R}^n$ correspond to the point $(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in $\mathbb{R}^{m+n}$. Now let A and B be compact subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. We shall use the Bolzano–Weierstrass theorem (Theorem 13.2.5) to show that $A \times B$ is compact.

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Let $\{(\mathbf{a}_k, \mathbf{b}_k)\}_k$ be a sequence in $A \times B$. Since A is compact, the sequence $\{\mathbf{a}_k\}_k$ has a convergent subsequence $\{\mathbf{a}_{k_j}\}_j$, and since B is compact, $\{\mathbf{b}_{k_j}\}_j$ has a convergent subsequence $\{\mathbf{b}_{k_{j_i}}\}_i$. Let $\mathbf{a}'_i = \mathbf{a}_{k_{j_i}}$ and $\mathbf{b}'_i = \mathbf{b}_{k_{j_i}}$. Then $\{(\mathbf{a}'_i, \mathbf{b}'_i)\}_i$ is a subsequence of $\{(\mathbf{a}_k, \mathbf{b}_k\}_k$, and the sequences $\{\mathbf{a}'_i\}_i$ and $\{\mathbf{b}'_i\}_i$ are both convergent. Since a sequence in $\mathbb{R}^p$ converges if and only if it converges componentwise (Theorem 13.2.1), it follows that $\{(\mathbf{a}'_i, \mathbf{b}'_i)\}_i$ is convergent. Hence, $A \times B$ is compact.

13.3

13.3.7 Let $A \subseteq S \subseteq \mathbb{R}^n$, and consider the following two statements:

(a) A is relatively closed in S.

(b) Whever a sequence $\{\mathbf{x}_k\}$ in A converges to a limit $\mathbf{x}_0 \in S$, then $\mathbf{x}_0 \in A$.

Claim: (a) $\iff$ (b).

Proof of $\Rightarrow$*:* Since A is relatively closed in S we have $A = S \cap F$, where F is closed in $\mathbb{R}^n$. Let $\{\mathbf{x}_k\}$ be a sequence in A that converges to $\mathbf{x}_0 \in S$. Since all $\mathbf{x}_k$ belong to A, they also belong to F, and because F is closed, Theorem 3.2.3 tells us that $\mathbf{x}_0 \in F$. Hence, $\mathbf{x}_0 \in S \cap F = A$.

Proof of $\Leftarrow$*:* Suppose that (b) is satisfied. We want to show that $A = S \cap F$ for some closed set F in $\mathbb{R}^n$. In fact, we shall show that $A = S \cap \bar{A}$, where $\bar{A}$ is the closure of A in $\mathbb{R}^n$. It is clear that $A \subseteq S \cap \bar{A}$, so what we need to show is that $S \cap \bar{A} \subseteq A$. Let $\mathbf{x}_0 \in S \cap \bar{A}$. Since $\mathbf{x}_0 \in \bar{A}$ there exists a sequence $\{\mathbf{x}_k\}$ of points in A converging to $\mathbf{x}_0$. Since $\mathbf{x}_0 \in S$, we have $\mathbf{x}_0 \in A$ by assumption. It follows that $S \cap \bar{A} \subseteq A$.

Now that we have proved the equivalence of (a) and (b) above, let us use it to prove part (b) of Theorem 13.3.5: Let $S \subseteq \mathbb{R}^n$ and let $\mathbf{f}$ be a function from S to $\mathbb{R}^m$. Then

$$\mathbf{f} : S \subseteq \mathbb{R}^n \to \mathbb{R}^m \text{ is continuous} \iff \mathbf{f}^{-1}(F) \text{ is relatively closed in } S \text{ for each closed set } F \text{ in } \mathbb{R}^m$$

Proof of $\Rightarrow$*:* Suppose that $\mathbf{f}$ is continuous and let F be a closed set in $\mathbb{R}^m$. We want to prove that $\mathbf{f}^{-1}(F)$ is relatively closed in S. By the equivalence (a) $\iff$ (b) above it suffices to show that, if a point $\mathbf{x}_0$ in S is the limit of a sequence $\{\mathbf{x}_k\}_k$ in $\mathbf{f}^{-1}(F)$ then $\mathbf{x}_0 \in \mathbf{f}^{-1}(F)$. If we have such a point $\mathbf{x}_0$ and such a sequence $\{\mathbf{x}_k\}$, then all $\mathbf{f}(\mathbf{x}_k) \in F$, and because $\mathbf{f}$ is continuous, $(\mathbf{x}_0) = \lim_k (\mathbf{x}_k)$. Thus, $\mathbf{f}(\mathbf{x}_0)$ the limit of a sequence in F, and since F is closed, $\mathbf{f}(\mathbf{x}_0) \in F$. Therefore $\mathbf{x}_0 \in \mathbf{f}^{-1}(F)$.

Proof of $\Leftarrow$*:* Suppose that $\mathbf{f}^{-1}(F)$ is relatively closed in S for every closed F in $\mathbb{R}^m$, and let $\mathbf{x}_0$ be a point in S. We want to show that $\mathbf{f}$ is continuous at $\mathbf{x}_0$. In other words, we want to show that $\mathbf{f}(\mathbf{x}_k) \to \mathbf{f}(\mathbf{x}_0)$ for every sequence $\{\mathbf{x}_k\}$ in S that converges to $\mathbf{x}_0$.

Suppose $(*)$ $\mathbf{x}_k \to \mathbf{x}_0$ but $\mathbf{x}_k \not\to \mathbf{f}(\mathbf{x}_0)$. Then there is an $\varepsilon > 0$ such that $\|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{x}_0)\| \geq \varepsilon$ for infinitely many k. Let $F = \{\mathbf{y} \in \mathbb{R}^m : \|\mathbf{y} - \mathbf{f}(\mathbf{x}_0)\| \geq \varepsilon\}$ be the complement in $\mathbb{R}^m$ of the open ε-ball around $\mathbf{f}(\mathbf{x}_0)$, and let $\{\mathbf{x}'_j\} = \{\mathbf{x}_{k_j}\}$ be the subsequence of $\{\mathbf{x}_k\}$ where $k_1 < k_2 \cdots$ run through all those k for which $\|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{x}_0)\| \geq \varepsilon$. The set F is closed in $\mathbb{R}^m$, and $\{\mathbf{x}'_j\}$ is a sequence in $\mathbf{f}^{-1}(F)$, the inverse image of F, with $\lim_{j\to\infty} \mathbf{x}'_j = \mathbf{x}_0 \in S$. By assumption, $\mathbf{f}^{-1}(F)$ is relatively closed in S, and by the equivalence (a) $\iff$ (b) above we must have $\mathbf{x}_0 \in \mathbf{f}^{-1}(F)$. But then $\mathbf{f}(\mathbf{x}_0) \in F$, and by the definition of F we must have $\|\mathbf{f}(x_0) - \mathbf{f}(\mathbf{x}_0)\| \geq \varepsilon > 0$, which is absurd. This shows that the assumption $(*)$ must be false, and so $\mathbf{f}$ is indeed continuous.

13.3.8 Assume first that $\mathbf{f}$ is continuous at $\mathbf{x}^0$. We shall prove that the defender can always win in this case. Let the challenger choose $\varepsilon > 0$. Since $\mathbf{f}$ is continuous at $\mathbf{x}^0$ the defender is able to choose a $\delta > 0$ such that $\|\mathbf{f}(\mathbf{x}) - \mathbf{x}^0)\| < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{x}^0\| < \delta$, and then the challenger will be unable to find an $\mathbf{x}$ that lets him win. Thus the defender wins.

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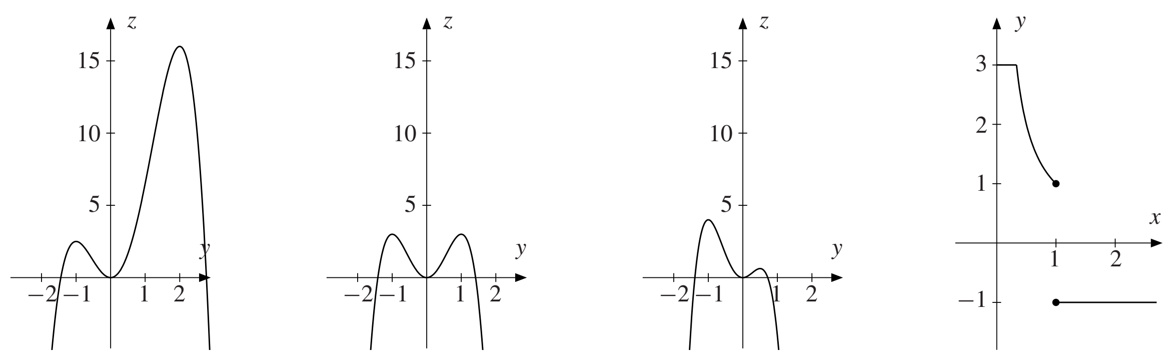
Now assume that $\mathbf{f}$ is discontinuous at $\mathbf{x}^0$. Then there will exist at least one $\varepsilon > 0$ that cannot be matched by any δ. So let the challenger choose such an ε. Then no matter what $\delta > 0$ the defender chooses, the challenger will be able to find an $\mathbf{x}$ with $\|\mathbf{x} - \mathbf{x}^0\| < \delta$ and $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)\| \geq \varepsilon$. Thus, in this case, the challenger wins.

13.4

13.4.3 For a fixed x, the maximum of $f(x, y)$ with respect to y for y in $[-3, 3]$ must be attained at a point where $f_2'(x, y) = 0$ or $y = \pm 3$. Since $f_2'(x, y) = -12xy^3 - 12(x-1)y^2 + 12y = -12xy(y-1/x)(y+1)$, $f(x, y)$ is strictly increasing with respect to y when $y \in (-\infty, -1]$, strictly decreasing in $[-1, 0]$, strictly increasing again in $[0, 1/x]$, and strictly decreasing in $[1/x, \infty)$. Hence the only possible maximum points are $y = -1$ and $y = 1/x$ if $x > 1/3$ and $y = -1$ and $y = 3$ if $0 < x \leq 1/3$. Simple calculations give $f(x, -1) = x + 2$, $f(x, 1/x) = (2x + 1)/x^3$, and $f(x, 3) = 162 - 351x$. It follows that $f(x, 1/x) - f(x, -1) = (x - 1)(x + 1)^3/x^3$, so for $x > 1$ the maximum occurs for $y = -1$, if $x = 1$ the maximum occurs for $y = \pm 1$, and if $1/3 < x \leq 1$ the maximum occurs for $y = 1/x$. Finally, if $0 < x \leq 1/3$, the maximum occurs for $y = 3$. The value function V is given by

$$V(x) = \begin{cases} 162 - 351x & \text{if } 0 < x \leq 1/3 \\ (2x + 1)/x^3 & \text{if } 1/3 < x \leq 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

It is clear that V is continuous, because the one-sided limits of $V(x)$ at $x = 1/3$ are equal, and so are the one-sided limits at $x = 1$. Figures M13.4.3(a)–(c) shows the graph of the function $y \mapsto f(x, y)$ for three different values of x, Fig. M13.4.3(d) shows the set $M(x)$ of maximizers as a function of x. Except at $x = 1$, the graph is the graph of a continuous function, as it should be because the maximizer is unique for each $x \neq 1$.



(a) $z = f(1/2, y)$ **(b)** $z = f(1, y)$ **(c)** $z = f(2, y)$ **(d)** $x \mapsto M(x)$

M13.4.3

13.5

13.5.7 Let S be a compact set in $\mathbb{R}^n$. Carathéodory's theorem (Theorem 13.5.1) tells us that every point in $\text{co}(S)$ can be written as a convex combination of at most $n + 1$ points in S. We claim that every point in $\text{co}(S)$ can be written as a linear combination of *exactly* $n + 1$ points in S. Indeed, if $\mathbf{x} = \lambda_1\mathbf{x}_1 + \cdots \lambda_m\mathbf{x}_m$ with $m < n + 1$, we just add $n + 1 - m$ terms that are all equal to $0\mathbf{x}_1$.

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Now let $T = \Delta^n \times S^{n+1}$, where $\Delta^n = \{(\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}^{n+1} : \lambda_i \geq 0 \text{ for } i = 1, \dots, n+1;\ \lambda_1 + \cdots + \lambda_{n+1} = 1\}$ and $S^{n+1} = S \times \cdots \times S$ is the Cartesian product of $n+1$ copies of S. Define a function $f : T \to \mathbb{R}^n$ by $f(\lambda_1, \dots, \lambda_{n+1}, \mathbf{x}_1, \dots, \mathbf{x}_{n+1}) = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_{n+1}\mathbf{x}_{n+1}$. Then f maps every point in T to a point in $\text{co}(S)$. On the other hand, the argument above shows that every point in $\text{co}(S)$ belongs to the image $f(T)$ of T under f, so in fact $\text{co}(S) = f(T)$. The set Δ^n is obviously a compact (i.e. closed and bounded) subset of $\mathbb{R}^{n+1}$, so if S is also compact, then so is T. (It follows from Problem 13.2.6 that, if A and B are compact subsets of $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, then $A \times B$ is a compact subset of $\mathbb{R}^{m+p}$. This immediately extends to the Cartesian product of a finite number of sets.) Since f is continuous, Theorem 13.3.3 shows that $\text{co}(S) = f(T)$ is compact.

What we have shown here is that, if S is closed and bounded, then so is $\text{co}(S)$. If S is closed but *unbounded*, then $\text{co}(S)$ need not even be closed. Consider, for example, the closed set $S = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy \geq 1\} \cup \{(0, 0)\}$. The convex hull of S is $\text{co}(S) = \{(x, y) : x > 0, y > 0\} \cup \{(0, 0)\}$, i.e. the open first quadrant together with the origin. This set is not closed. (Draw a picture!)

13.6

13.6.3 If $\mathbf{x}$ is not an interior point of the convex set S ($\subseteq \mathbb{R}^n$), then by Theorem 13.6.2 there exists a nonzero vector $\mathbf{a}$ in $\mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{z} \leq \mathbf{a} \cdot \mathbf{x}$ for every $\mathbf{z}$ in S. Then $S \subseteq H_- = \{\mathbf{z} : \mathbf{a} \cdot \mathbf{z} \leq \mathbf{a} \cdot \mathbf{x}\}$. Since the half space H_- is closed, we also have $\bar{S} \subseteq H_-$. Every open ball around $\mathbf{x}$ contains points that do not belong to H_-, for instance points of the form $\mathbf{x} + t\mathbf{a}$ for small positive numbers t. Hence, no open ball around $\mathbf{x}$ is contained in $\bar{S}$, and so S is not an interior point of $\bar{S}$.

14 Correspondences and Fixed Points

Proof of Theorem 14.1.5(c): We will use the result in (14.1.8) to prove that $H = G \circ F$ is lower hemicontinuous. Let $\mathbf{z}^0 \in H(\mathbf{x}^0)$ and let U be a neighbourhood of $\mathbf{z}^0$. Then $\mathbf{z}^0 \in G(\mathbf{y}^0)$ for some $\mathbf{y}^0$ in $F(\mathbf{x}^0)$. Since G is l.h.c. at $\mathbf{y}^0$, there exists a neighbourhood V of $\mathbf{y}^0$ such that $U \cap G(\mathbf{y}) \neq \emptyset$ for all $\mathbf{y}$ in V. Moreover, since $\mathbf{y}^0 \in F(\mathbf{x}^0)$, there is a neighbourhood N of $\mathbf{x}^0$ such that $V \cap F(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in N$. Let $\mathbf{x} \in N$ and $\mathbf{y} \in V \cap F(\mathbf{x})$. Then $U \cap G(\mathbf{y}) \neq \emptyset$, and if $\mathbf{z} \in U \cap G(\mathbf{y})$, then $\mathbf{z} \in G(\mathbf{y})$, $\mathbf{y} \in F(\mathbf{x})$, so $\mathbf{z} \in H(\mathbf{x})$.

14.1

14.1.4 We shall use the characterization of lower hemicontinuity in (14.1.8) on page 506 to show that if F and G are l.h.c. at $\mathbf{x}_0$, then so is H. Let U be an open neighbourhood of a point $(\mathbf{y}^0, \mathbf{z}^0)$ in $H(\mathbf{x}^0)$. Then there are open neighbourhoods U_1 and U_2 of $\mathbf{y}^0$ and $\mathbf{z}^0$ in $\mathbb{R}^l$ and $\mathbb{R}^m$, respectively, such that $U_1 \times U_2 \subseteq U$. Since F is l.h.c., there are neighbourhoods N_1 and N_2 of $\mathbf{x}^0$ such that $F(\mathbf{x}) \cap U_1 \neq \emptyset$ for all $\mathbf{x}$ in $N_1 \cap X$ and $G(\mathbf{x}) \cap U_2 \neq \emptyset$ for all $\mathbf{x}$ in $N_2 \cap X$. Let $N = N_1 \cap N_2$. Then $H(\mathbf{x}) \cap U \supseteq H(\mathbf{x}) \cap (U_1 \times U_2) = (F(\mathbf{x}) \cap U_1) \times (G(\mathbf{x}) \cap U_2) \neq \emptyset$ for all $\mathbf{x}$ in $N \cap X$. It follows that H is l.h.c.

For the result about upper hemicontinuity we need to assume that $F(\mathbf{x}^0)$ and $G(\mathbf{x}^0)$ are compact. Then $H(\mathbf{x}^0) = F(\mathbf{x}^0) \times G(\mathbf{x}^0)$ is also compact. Suppose that F and G are u.h.c. at $\mathbf{x}^0$. Since $F(\mathbf{x}^0)$ is compact, it is closed, and Note 14.1.1 then says that F has the closed graph property at $\mathbf{x}_0$. Because $F(\mathbf{x}^0)$ is bounded there exists a bounded and open set U in $\mathbb{R}^l$ that contains $F(\mathbf{x}^0)$, and since F is upper hemicontinuous at $\mathbf{x}^0$ there is a neighbourhood N_F of $\mathbf{x}^0$ such that $F(\mathbf{x}) \subseteq U$ for all $\mathbf{x}$ in N_F. Similarly, G has the closed graph property at $\mathbf{x}_0$ and there exists a bounded and open set V in $\mathbb{R}^m$ and a neighbourhood N_G of $\mathbf{x}^0$ such that $G(\mathbf{x}) \subseteq V$ for all $\mathbf{x}$ in N_G.

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Since F and G have the closed graph property at $\mathbf{x}^0$, so has H. This is an easy consequence of the definition (14.1.4). We are now all set to use Theorem 14.1.2 to prove that H is u.h.c at $\mathbf{x}^0$. Let $W = U \times V \subseteq \mathbb{R}^{l+m}$ and let $N = N_F \cap N_G$. Then W is bounded in $\mathbb{R}^{l+m}$, N is a neighbourhood of $\mathbf{x}^0$, and $H(\mathbf{x}) = F(\mathbf{x}) \times G(\mathbf{x}) \subseteq W$ for all $\mathbf{x}$ in N, so H is locally bounded near $\mathbf{x}^0$. The desired conclusion follows.

14.1.6 We shall use the characterization of lower hemicontinuity that is given in (14.1.8) to prove that G is l.h.c. Let $\mathbf{y}^0 \in G(\mathbf{x}^0)$. Then $\mathbf{y}^0$ is a convex combination $\mathbf{y}^0 = \sum_{i=1}^k \lambda_i \mathbf{y}_i^0$ of points $\mathbf{y}_i^0$ in $F(\mathbf{x}^0)$. If U is a neighbourhood of $\mathbf{y}^0$, then U contains an open ball $B = B(\mathbf{y}^0; \varepsilon)$ for some $\varepsilon > 0$. For each $i = 1, \ldots, k$, there is a neighbourhood N_i of $\mathbf{x}$ such that $F(\mathbf{x}) \cap B(\mathbf{y}_i^0; \varepsilon) \neq \emptyset$ for all $\mathbf{x}$ in $N_i \cap X$. Let $N = N_1 \cap \cdots \cap N_k$ be the intersection of these neighbourhoods. Then for any $\mathbf{x}$ in N and every i there is at least one point $\mathbf{y}_i$ in $F(\mathbf{x}) \cap B(\mathbf{y}_i^0; \varepsilon)$. If we let $\mathbf{y} = \sum_i \lambda_i \mathbf{y}_i$, then $\mathbf{y} \in \operatorname{co}(F(\mathbf{x})) = G(\mathbf{x})$. Moreover, $d(\mathbf{y}, \mathbf{y}_0) = \|\mathbf{y} - \mathbf{y}_0\| = \|\sum_i (\lambda_i \mathbf{y}_i - \lambda_i \mathbf{y}_i^0)\| \leq \sum_i \lambda_i \|\mathbf{y}_i - \mathbf{y}_i^0\| < \sum_i \lambda_i \varepsilon = \varepsilon$, so $\mathbf{y} \in B(\mathbf{y}_0; \varepsilon) \subseteq U$. It follows that $G(\mathbf{x}) \cap U \neq \emptyset$. Hence G is l.h.c.

14.1.10 Every *constant* correspondence is both l.h.c. and u.h.c. But even if $a(x)$ and $b(x)$ are constant functions, complications may arise if one or both of the endpoints of the interval $(a((x), b(x))$ sometimes, but not always, belong to $F(x)$. Consider for example the correspondences C_1 and C_2 given by

$$C_1(x) = \begin{cases} [0, 1] & \text{if } x \leq 0, \\ [0, 1) & \text{if } x > 0, \end{cases} \qquad C_2(x) = \begin{cases} [0, 1] & \text{if } x < 0, \\ [0, 1) & \text{if } x \geq 0. \end{cases}$$

C_1 is u.h.c. everywhere, while C_2 is not u.h.c. at $x = 0$. (But both of them are l.h.c. everywhere.)

Every correspondence F that satisfies the conditions in the problem is l.h.c., provided only that its effective domain is open—i.e., if $F(x_0) \neq \emptyset$, then $F(x) \neq \emptyset$ for all x sufficiently close to x_0.

With non-constant $a(x)$ and $b(x)$ many complications arise in connection with upper hemicontinuity. A detailed study would take us too far afield, so let us concentrate on the four possibilities

$$F(x) = [a(x), b(x)], \quad G(x) = [a(x), b(x)), \quad H(x) = (a(x), b(x)], \quad K(x) = (a(x), b(x))$$

(for all x). F has a closed graph and is locally bounded, so Theorem 14.1.2 implies that F is u.h.c. The other three are usually not u.h.c., except in the constant case. For example, the correspondence $F(x) = (-1 - x^2, 1 + x^2)$ is not u.h.c. at $x = 0$, since the open set $U = (-1, 1)$ contains $F(x)$ for $x = 0$ but not for any other value of x, no matter how close to 0 it may be. In fact, F is not u.h.c. at any other point either.

14.1.11 The set $F(\mathbf{x})$ is compact for each $\mathbf{x}$ in X. By Problem 13.5.7, $G(\mathbf{x}) = \operatorname{co}(F(\mathbf{x}))$ is also compact for each $\mathbf{x}$. If $G(\mathbf{x}^0)$ is contained in an open set U, there exists an $\alpha > 0$ such that $G(\mathbf{x}^0) \subseteq U' \subseteq U$, where U' is the open "α-neighbourhood" of $G(\mathbf{x}^0)$ defined as

$$U' = B(G(\mathbf{x}^0); \alpha) = \left\{ \mathbf{y} \in \mathbb{R}^m : \text{there is a } \mathbf{y}' \text{ in } G(\mathbf{x}^0) \text{ with } \|\mathbf{y} - \mathbf{y}'\| < \alpha \right\}$$

This follows from the technical result below, with $S = \complement U = \mathbb{R}^m \setminus U$ and $K = F(\mathbf{x}^0)$. Since F is u.h.c. and $F(\mathbf{x}^0) \subseteq G(\mathbf{x}^0) \subseteq U'$, there exists a $\delta > 0$ such that $F(\mathbf{x}) \subseteq U'$ for every $\mathbf{x}$ in $N = B(\mathbf{x}^0; \delta)$. It is not hard to see that U' is convex, and therefore $G(\mathbf{x}) = \operatorname{co}(F(\mathbf{x}))$ is also contained in U' (and so in U) for all $\mathbf{x}$ in N.

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A little technical result:

If S and K are disjoint closed subsets of $\mathbb{R}^m$ and K is compact, then there exists a positive number α such that $d(\mathbf{x}, \mathbf{y}) \geq \alpha$ for all $\mathbf{x}$ in S and all $\mathbf{y}$ in K. In other words, the distance between a point in S and a point in K is never less than α.

Proof: Let $\mathbf{y}$ be a fixed point in $\mathbb{R}^n$ and let $\mathbf{x}$ be a point in S. By Problem 13.3.5, $h(\mathbf{x}) = d(\mathbf{x}, \mathbf{y})$ attains a minimum at some point $\mathbf{x}^0$ in S. Let us call $h(\mathbf{x}^0)$ the *distance* between the point $\mathbf{y}$ and the set S, and denote it by $d(\mathbf{y}, S)$. If $\mathbf{y}'$ is another point in $\mathbb{R}^n$, then

$$d(\mathbf{y}', S) \leq d(\mathbf{y}', \mathbf{x}_0) \leq d(\mathbf{y}', \mathbf{y}) + d(\mathbf{y}, \mathbf{x}^0) = d(\mathbf{y}', \mathbf{y}) + d(\mathbf{y}, S)$$

so $d(\mathbf{y}', S) - d(\mathbf{y}, S) \leq d(\mathbf{y}', \mathbf{y})$. By symmetry, $d(\mathbf{y}, S) - d(\mathbf{y}', S) \leq d(\mathbf{y}, \mathbf{y}') = d(\mathbf{y}', \mathbf{y})$. Hence, $|d(\mathbf{y}', S) - d(\mathbf{y}, S)| \leq d(\mathbf{y}', \mathbf{y})$. It follows that $g(\mathbf{y}) = d(\mathbf{y}, S)$ is a continuous function of $\mathbf{y}$. (In the definition of continuity in (13.3.1), every $\varepsilon > 0$ can be matched by $\delta = \varepsilon$.) Since K is compact, g attains a minimum value α over K. Then $\alpha = g(\mathbf{y}^*) = d(\mathbf{y}^*, \mathbf{x}^*)$ for a point $\mathbf{y}^*$ in K and a point $\mathbf{x}^*$ in S, so $\alpha > 0$. ■

14.2

14.2.3 By Example 14.1.6, the budget correspondence $\mathcal{B}(\mathbf{p}, m)$ is lower hemicontinuous and has the closed graph property at any point $(\mathbf{p}^0, m_0)$ where $m_0 > 0$. It is also locally bounded near $(\mathbf{p}^0, m_0)$, so by Theorem 14.1.2, $\mathcal{B}$ is also upper hemicontinuous at $(\mathbf{p}^0, m_0)$. What if $m_0 = 0$? In that case $\mathcal{B}(\mathbf{p}^0, m_0) = \mathcal{B}(\mathbf{p}^0, 0)$ consists of a single point, namely the origin $\mathbf{0}$ in $\mathbb{R}^n$. If U is an open set in $\mathbb{R}^n$ that contains $\mathbf{0}$, then it will obviously contain $\mathcal{B}(\mathbf{p}, m)$ for any $(\mathbf{p}, m)$ close to $(\mathbf{p}^0, 0)$, and it follows that $\mathcal{B}$ is upper hemicontinuous at $(\mathbf{p}^0, 0)$. Lower hemicontinuity at that point follows easily from the result in (14.1.8).

Thus, $\mathcal{B}(\mathbf{p}, m)$ is continuous at every point $(\mathbf{p}, m)$ in $X = \mathbb{R}^n_{++} \times \mathbb{R}_+$.

The maximum theorem (Theorem 14.2.1) then implies that the demand correspondence $\xi(\mathbf{p}, m)$ is upper hemicontinuous and the indirect utility function $V(\mathbf{p}, m)$ is continuous. (Note that $\mathbf{x} \mapsto\!\!\!\!\rightarrow F(\mathbf{x})$ in Theorem 14.2.1 corresponds to $(\mathbf{p}, m) \mapsto\!\!\!\!\rightarrow \mathcal{B}(\mathbf{p}, m)$ in this problem, while $\mathbf{y}$ and $f(\mathbf{x}, \mathbf{y})$ in the theorem correspond to $\mathbf{x}$ and $U(\mathbf{x})$ in the problem.) The demand correspondence will not always be lower hemicontinuous.

Suppose that U is quasiconcave. Then, if $\mathbf{x}^1$ and $\mathbf{x}^2$ are distinct points in $\xi(\mathbf{p}, m)$ (i.e. maximum points for U over $\mathcal{B}(\mathbf{p}, m)$), Theorem 2.5.1 implies that $\lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$ also belongs to $\xi(\mathbf{p}, m)$ for every λ in $[0, 1]$. In other words, $\xi(\mathbf{p}, m)$ is a convex set.

A Sets, Completeness, and Convergence

A.1

A.1.3 The answer in the book has been pared down to the bone, with a function defined on a set with only two elements in it. For an example with a little more flesh on it, consider the following:

Let $f(x) = x^2$ for all x in $\mathbb{R}$, and define two intervals $S_1 = [-4, 2]$ and $S_2 = [-1, 3]$. The intersection of S_1 and S_2 is $S_1 \cap S_2 = [-1, 2]$. The images of S_1 and S_2 under f are $f(S_1) = [0, 16]$ and $f(S_2) = [0, 9]$, and the image of $S_1 \cap S_2$ is $f(S_1 \cap S_2) = [0, 4]$. Here, then, is a case where $f(S_1 \cap S_2)$ is a proper subset of $f(S_1) \cap f(S_2)$.

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A.1.4 A relation is a linear ordering if and only if it is (i) reflexive, (ii) transitive, (iii) anti-symmetric, and (iv) complete. For each of these four properties it is easy to see that if a relation R has that property, then so has R^{-1}. Let us just see how to handle (ii):

We are assuming that R is transitive relation in a set S. Let x, y, and z be elements in S such that $xR^{-1}y$ and $yR^{-1}z$. We want to show that $xR^{-1}z$. We have yRx and zRy, and since R is transitive, zRx. Therefore $xR^{-1}z$, as promised.

A.2

A.2.2 It is shown in the answer in the book that $s < r$, and since s is positive and $s^2 > 2$, we also have $s > 2$. Thus the rational number s is closer to $\sqrt{2}$ than r is. But how did anyone come up with the expression $s = (2 + r^2)/2r$ in the first place? The answer is Newton's method, which is a famous procedure for finding and improving approximate solutions to an equation $f(x) = 0$, where f is a differentiable function.

Given one approximate solution, x_0, we let $x_1 = x_0 - f(x_0)/f'(x_0)$, and then we construct x_2 from x_1 in the same fashion, and so on. If the initial approximation x_0 is good enough, the sequence $x_1, x_2, \dots$ will usually converge to a root of $f(x) = 0$. (A brief discussion of this method can be found in EMEA, for instance.) Now apply this procedure to the equation $f(x) = x^2 - 2 = 0$. If $x_0 = r$ is a positive number, then $x_1 = r - f(r)/f'(r) = r - (r^2 - 2)/2r = (r^2 + 2)/2r$ is precisely the number s.

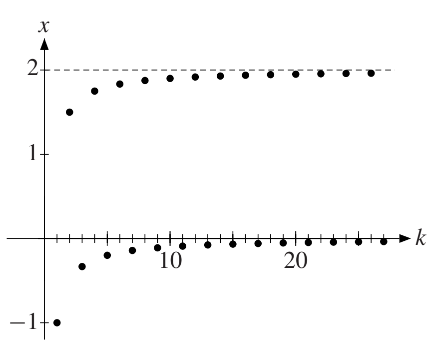
A.3

A.3.4 For the solution of this problem we need the fact that, if two convergent sequences have infinitely many terms in common, they must have the same limit. (This is an easy consequence of Theorem A.3.3.)

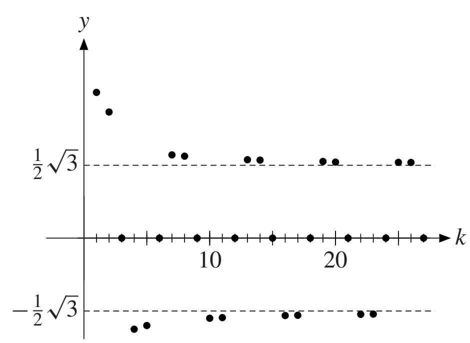
Consider the particular subsequences $\{w_k\} = \{x_{2k-1}\}$ and $\{z_k\} = \{x_{2k}\}$ of $\{x_k\}$. They converge to 0 and 2, respectively. Together, these two subsequences contain all the terms of the original sequence, so any subsequence of $\{x_k\}$ must have an infinite number of terms in common with at least one of $\{w_k\}$ and $\{z_k\}$. Hence, any *convergent* subsequence of $\{x_k\}$ must converge to either 0 or 2.

The six subsequences $\{y_{6k}\}$, $\{y_{6k+1}\}$, $\{y_{6k+2}\}$, $\{y_{6k+3}\}$, $\{y_{6k+4}\}$, and $\{y_{6k+5}\}$ converge to $\sin 0 = 0$, $\sin(\pi/3) = \frac{1}{2}\sqrt{3}$, $\sin(2\pi/3) = \frac{1}{2}\sqrt{3}$, $\sin(3\pi/3) = 0$, $\sin(4\pi/3) = -\frac{1}{2}\sqrt{3}$, and $\sin(5\pi/3) = -\frac{1}{2}\sqrt{3}$, respectively, and they contain all but the first five elements of $\{y_k\}$. In the same way as for $\{x_k\}$, it follows that any convergent subsequence of $\{y_k\}$ must converge to 0, $\frac{1}{2}\sqrt{3}$, or $-\frac{1}{2}\sqrt{3}$.

Figures MA.3.4(a) and MA3.4(b) illustrate the behaviour of $\{x_k\}$ and $\{y_k\}$.



MA.3.4(a)



MA.3.4(b)

A.3.5 (b) (i): For each natural number n, let $M_n = \sup\{x_k : k \geq n\}$ and $N_n = \sup\{y_k : k \geq n\}$. Then for all $k \geq n$, we have $x_k \leq M_n$ and $y_k \leq N_n$, and so $x_k + y_k \leq M_n + N_n$. Thus $\sup\{x_k + y_k : k \geq n\} \leq M_n + N_n \leq$

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$\underline{\lim}_{n\to\infty}(M_n + N_n) = \underline{\lim}_{n\to\infty} M_n + \underline{\lim}_{n\to\infty} N_n$, or $\overline{\lim}_{k\to\infty}(x_k + y_k) \le \overline{\lim}_{k\to\infty} x_k + \overline{\lim}_{k\to\infty} y_k$.
The proof of (ii) is similar and is left to the reader.

A.3.6 Let n and m be natural numbers with $n > m$, and let $p = n - m$. Then

$$\begin{aligned}
|x_n - x_m| &= |x_{m+p} - x_m| \\
&= |(x_{m+p} - x_{m+p-1}) + (x_{m+p-1} - x_{m+p-2}) + \cdots + (x_{m+2} - x_{m+1}) + (x_{m+1} - x_m)| \\
&\le |x_{m+p} - x_{m+p-1}| + |x_{m+p-1} - x_{m+p-2}| + \cdots + |x_{m+2} - x_{m+1}| + |x_{m+1} - x_m| \\
&< \frac{1}{2^{m+p-1}} + \frac{1}{2^{m+p-2}} + \cdots + \frac{1}{2^{m+1}} + \frac{1}{2^m} \\
&= \frac{1}{2^m}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-1}}\right) = \frac{1}{2^m}\frac{1 - (\frac{1}{2})^p}{1 - \frac{1}{2}} < \frac{1}{2^m}\frac{1}{1 - \frac{1}{2}} = \frac{1}{2^{m-1}}
\end{aligned}$$

which obviously becomes arbitrarily small for m sufficiently large.

B Trigonometric Functions

B.1

B.1.4 $\cos(y - \pi/2) = \sin y$ follows directly from Problem 3. Then, from the hints in the question, as well as (B.1.8) and the result of Problem 3 again, it follows that

$$\begin{aligned}
\sin(x + y) = \cos(x + y - \pi/2) &= \cos x \cos(y - \pi/2) - \sin x \sin(y - \pi/2) \\
&= \cos x \sin y + \sin x \cos y = \sin x \cos y + \cos x \sin y
\end{aligned}$$

Substituting $-y$ for y then yields $\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y) = \sin x \cos y - \cos x \sin y$.

B.1.6 It is clear from the definitions of $\sin x$ and $\cos x$ that $\sin(x + \pi) = -\sin x$ and $\cos(x + \pi) = -\cos x$ for all x. This also follows from Problem 4 and formula (B.1.8) together with the fact that $\sin \pi = 0$ and $\cos \pi = -1$. The formula for $\sin(x - y)$ in Problem 4 also shows that $\sin(\pi - x) = \sin x$ for all x. These results come in handy in this problem.

(a) $\sin(\pi - \pi/6) = \sin(\pi/6) = 1/2$.

(b) $(\cos(\pi + \pi/6) = -\cos(\pi/6) = -\frac{1}{2}\sqrt{3}$.

(c) By Problem 2(a), $\sin(-3\pi/4) = -\sin(3\pi/4) = -\frac{1}{2}\sqrt{2}$.

(d) $\cos(5\pi/4) = \cos(\pi/4 + \pi) = -\cos(\pi 4) = -\frac{1}{2}\sqrt{2}$.

(e) By formula (6), $\tan(7\pi/6) = \tan(\pi/6) = \frac{1}{3}\sqrt{3}$.

(f) $\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4) = \frac{1}{4}(\sqrt{6} - \sqrt{2})$.

B.1.7 (a) The formula for $\sin(x + y)$ in Problem 4 gives $\sqrt{2}\sin(x + \pi/4) - \cos x = \sqrt{2}(\sin x \cos(\pi/4) + \cos x \sin(\pi/4)) - \cos x = \sqrt{2}(\sin x \cdot \frac{1}{2}\sqrt{2} + \cos x \cdot \frac{1}{2}\sqrt{2}) - \cos x = \sin x$

(b) Since $\sin(\pi - x) = \sin x$ and $\cos(2\pi - x) = \cos(-x) = \cos x$, we have

$$\frac{\sin[\pi - (\alpha + \beta)]}{\cos[2\pi - (\alpha + \beta)]} = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \tan(\alpha + \beta)$$

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(c) Formula (B.1.8) and the formulas in Problems 3 and 4 give

$$\begin{aligned}\frac{\sin(a+x)-\sin(a-x)}{\cos(a+x)-\cos(a-x)} &= \frac{\sin a\cos x+\cos a\sin x-\sin a\cos x+\cos a\sin x}{\cos a\cos x-\sin a\sin x-\cos a\cos x-\sin a\sin x}\\ &= \frac{2\cos a\sin x}{-2\sin a\sin x} = -\frac{\cos a}{\sin a} = -\cot a\end{aligned}$$

B.1.8 With $x = \frac{1}{2}(A+B)$ and $y = \frac{1}{2}(A-B)$ we get $\sin A - \sin B = \sin(x+y) - \sin(x-y) = \sin x\cos y + \cos x\sin y - \sin x\cos y + \cos x\sin y = 2\cos x\sin y = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$.

B.1.12 (a) This is a sine curve $y = A\sin(ax)$ with $A = 2$ and $a(8\pi) = 2\pi$, i.e. $a = 1/4$.

(b) $y = 2 + \cos x$. (A cosine curve with amplitude 1 and period 2π shifted 2 units upwards.)

(c) $y = 2e^{-x/\pi}\cos x$. (An exponentially damped cosine curve with amplitude $2e^{-x/\pi}$.)

B.1.13 Since the lengths of the line segments AC and BD are equal, we have $(\cos x - \cos y)^2 + (\sin x + \sin y)^2 = (\cos(x+y) - 1)^2 + \sin^2(x+y)$. The left-hand side is

$$\begin{aligned}\text{LHS} &= \cos^2 x - 2\cos x\cos y + \cos^2 y + \sin^2 x + 2\sin x\sin y + \sin^2 y\\ &= 2 - 2\cos x\cos y + 2\sin x\sin y\end{aligned}$$

and the right-hand side is

$$\text{RHS} = \cos^2(x+y) - 2\cos(x+y) + 1 + \sin^2(x+y) = 2 - 2\cos(x+y)$$

where we have repeatedly used that $\sin^2 u + \cos^2 u = 1$. The equation LHS = RHS implies that $\cos(x+y) = \cos x\cos y - \sin x\sin y$.

B.2

B.2.1 (b) $(x\cos x)' = x'\cos x + x(\cos x)' = \cos x - x\sin x$.

(c) Let $u = x^2$. Then $\dfrac{d}{dx}(\tan(x^2)) = \dfrac{d}{du}(\tan u)\dfrac{du}{dx} = \dfrac{1}{\cos^2 u}2x = \dfrac{2x}{\cos^2(x^2)}$.

(d) $(e^{2x}\cos x)' = (e^{2x})'\cos x + e^{2x}(\cos x)' = 2e^{2x}\cos x - e^{2x}\sin x = e^{2x}(2\cos x - \sin x)$.

B.2.4 (c) All you need is the chain rule and a steady hand.

B.2.5 (c) $\displaystyle\lim_{t\to 0}\frac{1-\cos t}{t^2} = \frac{\text{“}0\text{”}}{0} = \lim_{t\to 0}\frac{\sin t}{2t} = \frac{1}{2}\lim_{t\to 0}\frac{\sin t}{t} = \frac{1}{2}$ by (B.2.10) (or just use l'Hôpital's rule again).

B.2.6 The derivative of $f'(x)$ is $f'(x) = 3(\sin x - x - 1)^2(\cos x - 1)$. It is easy to see that $\sin x < x + 1$ for all $x > 0$, because $\sin x \le 1 < x + 1$. Moreover, $\cos x < 1$ for all x in the open interval $J = (0, 3\pi/2)$. It follows that $f'(x) < 0$ for all x in J. Thus, f is strictly decreasing in the closed interval $I = [0, 3\pi/2]$, and attains its maximum value -1 at $x = 0$ and its minimum value $-(2 + 3\pi/2)^3 \approx -302.43$ at $x = 3\pi/2$.

B.2.8 You can read all these values off from Table B.2.1.

B.2.9 (a) Let $u = 2x$. The chain rule yields $\dfrac{d}{dx}(\arcsin 2x) = \dfrac{d}{du}(\arcsin u)\dfrac{du}{dx} = \dfrac{2}{\sqrt{1-u^2}} = \dfrac{2}{\sqrt{1-4x^2}}$.

(b) $(d/dx)(\arctan v)$ with $v = 1 + x^2$.

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(c) Let $w = \sqrt{x}$. Then $\dfrac{d}{dx}(\arccos\sqrt{x}\,) = \dfrac{d}{dw}(\arccos w)\dfrac{dw}{dx} = -\dfrac{1}{\sqrt{1-w^2}}\dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2\sqrt{1-x}\sqrt{x}}$.

B.2.10 (c) Integration by parts yields

$$\begin{aligned} I &= \int \sin^2 x\,dx = \int \sin x(-\cos x)'\,dx = \sin x(-\cos x) - \int (\sin x)'(-\cos x)\,dx \\ &= -\sin x\cos x + \int \cos^2 x\,dx = -\sin x\cos x + \int (1-\sin^2 x)\,dx \end{aligned}$$

Hence, $I = -\sin x\cos x + x - I + C$, and we get $I = \frac{1}{2}(x - \sin x\cos x) + C_1$, where $C_1 = \frac{1}{2}C$. Note: When integrating trigonometric functions it is very easy to get wrong signs here and there, and integration by parts also often leads to such mistakes, so it is a good rule to check the results by finding the derivatives when that is possible.

(d) Integration by parts here too.

B.2.11 (a) Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\displaystyle\int \tan x\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$.

(b) With $v = \sin x$, we get $dv = \cos x\,dx$ and $\displaystyle\int \cos x e^{\sin x}\,dx = \int e^v\,dv = e^v + C = e^{\sin x} + C$.

(c) As in part (a), let $u = \cos x$. Then $\displaystyle\int \cos^5 x\sin x\,dx = \int -u^5\,dx = -\tfrac{1}{6}u^6 + C = -\tfrac{1}{6}\cos^6 x + C$.

B.3

B.3.3 To simplify a complex fraction $(a+bi)/(c+di)$, where a, b, c, d are real, it is usually a good idea to multiply both the numerator and the denominator by $c - di$, the conjugate of the original denominator. This has the effect of making the denominator real (and positive) because $(c-di)(c+di) = c^2 - (di)^2 = c^2 - d^2i^2 = c^2 + d^2$. Thus,

(a) $\dfrac{(3+2i)(1+i)}{(1-i)(1+i)} = \dfrac{3+5i+2i^2}{1-i^2} = \dfrac{1+5i}{2} = \dfrac{1}{2} + \dfrac{5}{2}i$, (b) $\dfrac{(4-3i)(-i)}{i(-i)} = \dfrac{-3-4i}{1} = -3-4i$.

(c) Simplify the numerator and denominator before making the denominator real: $\dfrac{(3-2i)(2-i)}{(-1-i)(3+2i)} = \dfrac{6-7i+2i^2}{-3-5i-2i^2} = \dfrac{4-7i}{-1-5i} = \dfrac{(4-7i)(-1+5i)}{(-1)^2-(5i)^2} = \dfrac{-4+27i-35i^2}{1-25i^2} = \dfrac{31}{26} + \dfrac{27}{26}i$.

(d) $\dfrac{1-i}{1+i} = \dfrac{(1-i)^2}{1-i^2} = \dfrac{1-2i+i^2}{2} = \dfrac{-2i}{2} = -i$, so $\left(\dfrac{1-i}{1+i}\right)^3 = (-i)^3 = -i^3 = -i^2 i = i$.

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Corrections to FMEA 2nd edn (2008), first printing

(Spelling and grammatical mistakes are not included.)

Page 26, line 12 from the bottom. . . . (See Example B.3.2.)

Page 59, Theorem 2.3.5: In parts (b) and (c) the conclusion should be that $U(\mathbf{x}) = F(f(\mathbf{x}))$ is convex!

Page 117, Theorem 3.3.1(b): Add the assumption that S is convex.

Page 124, Problem 3.3.10: Replace the inequality in $(*)$ with equality.

Page 132, Theorem 3.5.1, line 1–2: Assume that f and $g_1, \dots, g_m$ are defined in a set S and that $\mathbf{x}^*$ is an interior point of S.

Page 164, Problem 10, line 5: The function f must also be continuous.

Page 192, Problem 6: The differential equation has no solution defined over the entire real line. We must be satisfied with a function x that satisfies the conditions in the problem in an open interval around 0.

Page 199, Problem 6, line 2: . . . all $t > 0$.

Page 225, Problem 6, line 3: . . . , provided $\dot{x} \neq 0$, we have

Page 352, Theorem 9.11.2, line 2: . . . that $\int_{t_0}^{\infty} |f(t, x(t), u(t))| e^{-t}\, dt < \infty$ for all . . .

Page 357, Problem 9.12.3(c), line 2: . . . with $K(t) > 0$ for

Page 382: The entries in the first column of the table should be "$t \in [0, t^*]$" and "$t \in (t^*, T]$".

Page 444, Problem 12.4.2(b): . . . in Theorem 12.4.1 are . . .

Page 454, formula (10): $E\big[F_2'(t, x_t, x_{t+1}(x_t, V_t), V_t) \,|\, v_{t-1}\big] + F_3'(t-1, x_{t-1}, x_t, v_{t-1}) = 0$

Page 455, line 5: . . . . Next, for $t = T$,

Page 455, line 8: $\frac{2}{3} v_{T-1}$ should be $\frac{2}{3} V_{T-1}$.

Page 457, Problem 12.6.6: The summation goes from $t = 0$, and the scrap value is $X_T^{1/2}$.

Page 508: See SM (not Problem 11) for the proof of Theorem 14.1.5(c).

Page 566, Problem 2.6.1(c): $f(x, y) \approx x + 2y - \frac{1}{2}x^2 - 2xy - 2y^2$

Page 566, Problem 2.6.4: $z \approx 1 - x + y + \frac{3}{2}x^2 - 2xy + \frac{1}{2}y^2$

Page 566, Problem 2.7.3, line 2: . . . , $w_x = 1/2$.

Page 570, Problem 3.7.3, line 2: $f^*(r, s) = \frac{1}{4}r^2$. . . . to find the squares of the largest

Page 570, Problem 3.8.3, line 2: Replace (c) by (b).

Page 570, Problem 3.8.5, line 1: Drop the labels (a) and (b).

Page 571: New Problem 3.11.1: See SM.

Page 574, Figure A5.2.2: The integral curve through (0, 2) is only the semicircle above the t-axis.

Page 576, Problem 5.6.1: Delete "and $x \equiv 0$". (We require $x > 0$.)

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Page 577, Problem 5.7.3, line 2: . . . . See Fig. A5.7.3(b).

Page 582, Problem 6.9.6: Drop the references to (a) and (b).

Page 583, Problem 7.2.3: Remove (b) from line 2.

Page 586, Problem 9.4.2: . . . where $A = 2e/(e^2 - 1)$.

Page 588, Problem 9.7.3: (a) $\ldots,\ x^*(t) = \dfrac{e^{(\alpha-2\beta)T+\alpha t}}{e^{(\alpha-2\beta)T} - 1} - \dfrac{e^{2(\alpha-\beta)t}}{e^{(\alpha-2\beta)T} - 1} = \dfrac{\big(e^{(\alpha-2\beta)T} - e^{(\alpha-2\beta)t}\big)e^{\alpha t}}{e^{(\alpha-2\beta)T} - 1},\ \ldots$

Page 589, Problem 9.11.4: $(x^*(t), u^*(t)) = \begin{cases} (e - e^{-t}, 1) & \text{if } t \in [-1, 0] \\ (e - 1, 0) & \text{if } t \in (0, \infty) \end{cases}, \qquad p(t) = e^{-t}.$

Page 589, Problem 9.12.3, line 1: . . . , and so $\dot{C}^*/C^* + \dot{\lambda}/\lambda = 0$. . . .

Page 590, Problem 10.4.3, line 3: . . . . When $u = 2$, $\dot{x} = -3$ and . . .

Page 591, Problem 10.6.3, line 2: Replace x_* by x^* and t^* by t'.

Page 593, Problem 11.7.2, line 1: (a) $x^* \approx -2.94753$.

Page 595, Problem 12.6.6: $u_t(x) = x_t/4a_t^2 = x_t/(1 + a_{t+1}^2)$, $a_T = a/2$, $a_t = \frac{1}{2}(1 + a_{t+1}^2)^{1/2}$ for $t < T$.

Page 595, Problem 12.6.8: $x_1 = \frac{1}{2} - v_0$, $x_2 = 1 - v_0 - v_1$, $x_3 = \frac{3}{2} - v_0 - v_1 - v_2$

Page 599, Problem 14.2.3: The demand correspondence ξ is upper hemicontinuous, but it need not be lower hemicontinuous.

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