Problem 39

(a) At stationary points, we must have

$$f_1'(x,y) = \frac{1}{x+y} - 2x + 1 = 0$$

$$f_2'(x,y) = \frac{1}{x+y} - 2y = 0$$

Subtracting the second equation from the first to cancell out $1/(x+y)$:

$$we\ get\ 2y = 2x - 1 \rightarrow y = x - \frac{1}{2}$$

Substituting this into the first FOC:

$$\frac{1}{2x - \frac{1}{2}} = 2x - 1 \rightarrow 1 = 4x^2 - 3x + \frac{1}{2} \rightarrow 4x^2 - 3x - \frac{1}{2} = 0$$

$$\rightarrow 8x^2 - 6x - 1 = 0.$$

$$x = \frac{6 \pm \sqrt{36+32}}{16} = \frac{6 \pm 2\sqrt{17}}{16} = \frac{3 \pm \sqrt{17}}{8}$$

$$since\ x > 0\ and \frac{3-\sqrt{17}}{8} < 0, we\ must\ have\ x^* = \frac{3+\sqrt{17}}{8}, then\ y^* = x^* - \frac{1}{2} = \frac{\sqrt{17}-1}{8}$$

$$so\ we\ find\ one\ stationary\ point\ (x^*, y^*) = (\frac{3+\sqrt{17}}{8}, \frac{\sqrt{17}-1}{8}).$$

(b) Simply put if we have

$$f_{11}''(x,y) \le 0, \quad f_{22}''(x,y) \le 0, \quad and$$

$$f_{11}''(x,y) f_{22}''(x,y) - \left(f_{12}''(x,y)\right)^2 \ge 0\ for\ all\ x > 0, y > 0$$

Then (x^*, y^*) is global maximum,

if

$$f_{11}''(x,y) \ge 0, \quad f_{22}''(x,y) \ge 0, \quad and$$

$$f_{11}''(x,y) f_{22}''(x,y) - \left(f_{12}''(x,y)\right)^2 \ge 0\ for\ all\ x > 0, y > 0.$$

Then (x^*, y^*) is global minimum.

Let's first look at the second-order derivatives:

$$f_{11}''(x,y) = -(x+y)^{-2} - 2 < 0, \quad f_{22}''(x,y) = -(x+y)^{-2} - 2 < 0$$

$$f_{12}''(x,y) = f_{21}''(x,y) = -(x+y)^{-2},$$

$$f_{11}''(x,y) f_{22}''(x,y) - \left(f_{12}''(x,y)\right)^2 = (-(x+y)^{-2} - 2)^2 - (-(x+y)^{-2})^2$$

$$= ((x+y)^{-2} + 2)^2 - ((x+y)^{-2})^2 = (x+y)^{-4} + 4(x+y)^{-2} + 4 - (x+y)^{-4}$$

$$= 4 + 4(x+y)^{-2} > 0\ since\ x > 0, y > 0.$$

$$So\ we\ have\ (x^*, y^*) = \left(\frac{3+\sqrt{17}}{8}, \frac{\sqrt{17}-1}{8}\right)\ which\ is\ an\ interior\ point\ for\ function$$

$$f(x,y)\ defined\ in\ a\ convex\ set\ x > 0, y > 0.$$

$$We\ also\ know\ that\ for\ all\ x > 0, y > 0, one\ has:$$

$$f_{11}''(x,y) \le 0, \quad f_{22}''(x,y) \le 0, \quad and$$

$$f''_{11}(x,y)f''_{22}(x,y) - \left(f''_{12}(x,y)\right)^2 \geq 0.$$
$$then\ (x^*, y^*)\ is\ a\ global\ maximum\ point\ of\ f.$$

Problem 138

(a)

$x + y + z = 1$ and $x^2 + y^2 + z^2 = 1$ form an nonempty, closed and bounded set through which $f(x, y, z)$ is continuous, so according to the Extreme Value Theorem there must exist both a maximum and a minimum in this set.

$$\mathcal{L}(x,y,z) = e^x + y + z - \lambda(x + y + z - 1) - \mu(x^2 + y^2 + z^2 - 1)$$
$$\mathcal{L}'_1(x,y,z) = e^x - \lambda - 2\mu x = 0$$
$$\mathcal{L}'_2(x,y,z) = 1 - \lambda - 2\mu y = 0$$
$$\mathcal{L}'_3(x,y,z) = 1 - \lambda - 2\mu z = 0$$
$$\mathcal{L}'_2(x,y,z) - \mathcal{L}'_3(x,y,z) = 2\mu(z - y) = 0$$
$$\rightarrow either\ \mu = 0\ or\ z - y = 0$$
$$if\ \mu = 0, we\ have\ that\ 1 - \lambda = 0\ so\ \lambda = 1, then\ e^x - \lambda = 0 \rightarrow x = 0.$$

The two constraints then become:

$$y + z = 1, \quad y^2 + z^2 = 1$$
$$\rightarrow y^2 + (1 - y)^2 = 2y^2 - 2y + 1 = 1 \rightarrow 2y^2 - 2y = 0$$
$$2y(y - 1) = 0 \rightarrow y = 0, z = 1\ or\ y = 1, z = 0.$$

We have then got two solution candidates

$$(x_1, y_1, z_1, \lambda_1, \mu_1) = (0, 1,0,1,0), \qquad f(x_1, y_1, z_1) = 2$$
$$and\ (x_2, y_2, z_2, \lambda_2, \mu_2) = (0,0,1,1,0), \qquad f(x_2, y_2, z_2) = 2.$$

$$if\ z = y, the\ two\ constraints\ could\ be\ rewritten\ as:$$
$$x + 2y = 1\ ,\ \ x^2 + 2y^2 = 1$$
$$Then\ y = \frac{1 - x}{2}, substituting\ into\ the\ second\ constraint\ gives\ us:$$
$$\rightarrow x^2 + 2\left(\frac{1 - x}{2}\right)^2 = x^2 + \frac{1}{2}(1 - 2x + x^2) = \frac{3}{2}x^2 - x + \frac{1}{2} = 1$$
$$\rightarrow 3x^2 - 2x - 1 = 0$$
$$\rightarrow (x - 1)(3x + 1) = 0$$
$$so\ we\ have\ x = 1, y = z = 0, or\ x = -\frac{1}{3}, y = z = \frac{2}{3}.$$

For $x = 1, y = z = 0$,

$$\mathcal{L}'_1(x,y,z) = e - \lambda - 2\mu = 0$$
$$\mathcal{L}'_2(x,y,z) = 1 - \lambda = 0$$
$$\rightarrow \lambda = 1, \mu = \frac{e - 1}{2}$$

For $x = -\frac{1}{3}, y = z = \frac{2}{3}$,

$$\mathcal{L}'_1(x,y,z) = e^{-\frac{1}{3}} - \lambda + \frac{2}{3}\mu = 0$$

$$\mathcal{L}_2'(x,y,z) = 1 - \lambda - \frac{4}{3}\mu = 0$$

$$\rightarrow \mu = \frac{1 - e^{-\frac{1}{3}}}{2}, \lambda = \frac{2e^{-\frac{1}{3}} + 1}{2}$$

We have then got another two solution candidates

$$(x_3, y_3, z_3, \lambda_3, \mu_3) = \left(1{,}0{,}0{,}1, \frac{e-1}{2}\right), \quad f(x_3, y_3, z_3) = e.$$

$$(x_4, y_4, z_4, \lambda_4, \mu_4) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2e^{-\frac{1}{3}} + 1}{2}, \frac{1 - e^{-\frac{1}{3}}}{2}\right), \quad f(x_4, y_4, z_4) = e^{-\frac{1}{3}} + \frac{4}{3} < 1 + \frac{4}{3} < e.$$

It is easy to see that solution candidate

$$(x_3, y_3, z_3, \lambda_3, \mu_3) = \left(1{,}0{,}0{,}1, \frac{e-1}{2}\right) \textit{ with } \quad f(x_3, y_3, z_3) = e$$

Solve the problem.

(b)

$$\max f(x,y) \;\; s.t. \;\; g(x,y) = c$$
$$\textit{if we write the maximized } f \textit{ as } f^*, \textit{then}$$
$$f^*(c + dc) - f^*(c) \approx \lambda(c)dc.$$

$$\Delta f^*(x,y,z) \approx \lambda(0.02) + \mu(-0.02) = 0.02 - 0.01(e-1)$$
$$= 0.01(3 - e)$$

Problem 86

(a) $\mathcal{L}(x,y,z) = x^2 + x + y^2 + z^2 - \lambda(x^2 + 2y^2 + 2z^2 - 16)$

$$\mathcal{L}_1'(x,y,z) = 2x + 1 - 2\lambda x = 0$$
$$\mathcal{L}_2'(x,y,z) = 2y - 4\lambda y = 0 = 2y(1 - 2\lambda)$$
$$\mathcal{L}_3'(x,y,z) = 2z - 4\lambda z = 0 = 2z(1 - 2\lambda)$$
$$\mathcal{L}_2'(x,y,z) - \mathcal{L}_3'(x,y,z) = 2(y - z)(1 - 2\lambda) = 0$$

If $1 - 2\lambda = 0 \rightarrow \lambda = \frac{1}{2}$

$$2x + 1 - x = 0 \rightarrow x = -1,$$
$$\textit{substituting this into the constraint gives us}:$$
$$1 + 2y^2 + 2z^2 = 16 \rightarrow y^2 + z^2 = \frac{15}{2}$$

So our first solution candidate(s) are:

$$(x,y,z) = (-1, y, z) \textit{ with } y^2 + z^2 = \frac{15}{2}, \qquad f(x,y,z) = \frac{15}{2}$$

If $\lambda \neq \frac{1}{2}, \textit{then } y = z = 0 \rightarrow x^2 = 16 \rightarrow x = \pm 4 \rightarrow \lambda = \frac{9}{8} \textit{ or } \frac{7}{8} \textit{ satisfies } \lambda \neq \frac{1}{2}$

So our second and third solution candidates are

$$(x,y,z) = (4{,}0{,}0), \qquad f(x,y,z) = 20 \quad \textit{and}$$

$$(x, y, z) = (-4,0,0), \qquad f(x, y, z) = 12.$$

Easy to see that $(x, y, z) = (4,0,0)$ *gives the maximum* $f(x, y, z) = 20$,

And $(x, y, z) = (-1, y, z)$ *with* $y^2 + z^2 = \frac{15}{2}$ *gives the minimum* $f(x, y, z) = \frac{15}{2}$

(b) $\mathcal{L}(x, y, z) = x^2 + x + y^2 + z^2 - \lambda(x^2 + 2y^2 + 2z^2 - 16)$

$$\mathcal{L}_1'(x, y, z) = 2x + 1 - 2\lambda x = 0$$

$$\mathcal{L}_2'(x, y, z) = 2y - 4\lambda y = 0 = 2y(1 - 2\lambda)$$

$$\mathcal{L}_3'(x, y, z) = 2z - 4\lambda z = 0 = 2z(1 - 2\lambda)$$

Introduce the complementary slackness condition:

$$\lambda \geq 0, with\ \lambda = 0\ if\ x^2 + 2y^2 + 2z^2 < 16$$

Let's first assume $x^2 + 2y^2 + 2z^2 = 16$, then we have $\lambda \geq 0$, and the problem is identical with (a), which gives us the following solution candidates (and for all these candidates we can verify that indeed $\lambda \geq 0$):

$$(x, y, z) = (-1, y, z)\ with\ y^2 + z^2 = \frac{15}{2}, \qquad f(x, y, z) = \frac{15}{2}$$

$$(x, y, z) = (4,0,0), \qquad f(x, y, z) = 20$$

$$(x, y, z) = (-4,0,0), \qquad f(x, y, z) = 12$$

Then we assume $x^2 + 2y^2 + 2z^2 < 16$, which means that $\lambda = 0$

The FOCs then gives us $y = z = 0, x = -\frac{1}{2}$.

So we have another solution candidates $(x, y, z) = \left(-\frac{1}{2}, 0,0\right)$ *with* $f(x, y, z) = -\frac{1}{4}$

Easy to see that $(x, y, z) = (4,0,0)$ *gives the maximum* $f(x, y, z) = 20$,

And $(x, y, z) = \left(-\frac{1}{2}, 0,0\right)$ *gives the minimum* $f(x, y, z) = -\frac{1}{4}$