UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Term paper in: ECON3120/4120 – Mathematics 2: Calculus and Linear Algebra

Handed out: 25.09.2017

Handed in: 09.10.17 – by 14.00 in Fronter.

Further instructions:

- This term paper is compulsory.
- Candidates who have passed the compulsory term paper in a previous semester, shall not hand in the term paper again.
- Candidates who have passed ECON3120/4120 before autumn 2016 and wish to re-take the exam need to get the term paper approved before they can re-take the exam.
- The grade of the term paper does not influence the final grade in the course.
- You must deliver the term paper within the given deadline. If the term paper is handed in after the deadline, we will not correct it. *
- You must hand in your term paper in Fronter.
- **Note:** You are allowed to discuss the term paper and how to solve it with each, but you have to formulate your own answers. Only single-authored papers are accepted, and papers that for all practical purposes are identical will not be approved.
- Information about citing and referring to sources: https://www.uio.no/english/studies/examinations/sources-citations/
- Information about consequences of cheating: https://www.uio.no/english/studies/examinations/cheating/
- If the term paper is not approved, you will be given a new attempt (this does not apply if you deliver a blank page). If the term paper is still not approved, you will not be permitted to sit the exam in this course.

*) If you believes that you have a valid reason not to meet the deadline, you must seek a formal extension from the Department of Economics. Read more about this here: http://www.uio.no/english/studies/admin/compulsory-activities/sv-absence-from-compulsory-tuition-activities.htmla

University of Oslo / Department of Economics **English only***

ECON3120/4120 Mathematics 2

Compulsory term paper, fall term 2017.

There are 2 pages of problems to be solved, not counting this page.

Justifying answers:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Minimum requirements to pass this assignment:

- You will pass if *each of problems 1, 2, and 3* is scored as good enough to pass, if it were judged to be one exam stand-alone.
  (The commonly applied pass mark in mathematics is forty percent, and this course by default uses uniform weighting over letter-enumerated items.)
- If you fail one of the three problems despite a decent attempt at it, we may still let you pass upon judging the overall quality of the paper. In particular, we shall take into account what parts are acknowledged as demanding.

The paper does *not* count towards your final course grade!

- Passing the term paper is required in order to sit in on the exam (see the Department's rules). Other than that, it does not in any way count towards your grade, and the exam grading committee will not see your term paper.

If you have questions concerning ...

- Questions concerning the problem set (the mathematics): ask the teachers.
- All other questions: `mailto:post@econ.uio.no`
- And, keep an eye on the course website – in particular the Messages section – in case of any clarifications or other information.

*Answers can be given in English, Norwegian, Swedish or Danish, as per the regulations for the exam.

Problem 1

(a) Let $h(x) = x^e \cdot [e^{sx} + e^{s^3 x}]$ and $H(x) = \frac{\ln(1+h(x))}{h(x)}$. Consider for each $s \neq 0$ the limits

$$\lim_{x\to 0^+} H(x) \qquad \text{and} \qquad \lim_{x\to+\infty} H(x)$$

For each $s \neq 0$:[†] what does l'Hôpital's rule tell you about each of these limits?

(b) For s in a certain range, the function $h(x)$ of part (a) will have a global extremum $x^* > 0$. The extreme value $V = h(x^*)$ depends on s. Find an expression for $V'(s)$.

(c) Let $u(x) = x - x^x$ and $v(x) = \log_{\sqrt{x}}(1 - u(x))$. Find $v'(x)$.

In the following, let $f(x, y) = y \cdot \left(e^{x^2 y} - x^2 y - 1\right) - 14$.

(d) In this part, you shall show *by contradiction* that $(0, 0)$ is a saddle point for f, *without* invoking second derivatives. Proceed as follows:

- First, you must show that it is a stationary point.
- Note that $f(x, y) = y \cdot w(x^2 y) - 14$ where $w(z) = e^z - 1 - z$ is > 0 for all $z \neq 0$. Use this to show that $f(x, y) - f(0, 0)$ has the same sign as y, as long as $xy \neq 0$.
- Assume for contradiction that $(0, 0)$ is a local maximum point, i.e., that we have $f(0, 0) - f(x, y) \geq 0$ for all (x, y) that are sufficiently close to $(0, 0)$. Point out that this contradicts the previous bullet item. Then do the analogous argument to disprove local minimum.

(e) Can the second-derivative test classify $(0, 0)$?

(f) Let from now on $t > 0$ be a constant, and consider the problem

$$\max\ f(x, y) \qquad \text{subject to} \qquad (tx + 1)^2 y \leq 1, \qquad x \geq 0 \qquad \text{and} \quad y \geq 0 \qquad \text{(P)}$$

- State the associated Kuhn–Tucker conditions, and explain why they are satisfied at $(0, 0)$.
- Do we know already, without further calculations, that problem (P) has a solution?

Problem 2 Compendium problem 28.[‡]

Note that there are partial answers given on page 36, but without the justification required for an exam nor for this term paper.

You are allowed to use those answers for an earlier letter item (e.g. "(b)") in a later letter item (e.g. "(d)") regardless of whether you managed to solve the earlier one.

[†] when it says "each $s \neq 0$", that means you must also check $s < 0$. Be warned that you can *not* expect this reminder on the exam!

[‡] Parts of it could be demanding – though certainly not so demanding to *pass* – and you might want to do the seminar-assigned problem 32 first.

Problem 3 Let r, s, t, u, v be real constants with $s > 0, t > 0, u > 0$. Define the matrices

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{pmatrix} s & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & s \end{pmatrix}$$

and denote by $\mathbf{I}$ the identity matrix and by $\mathbf{0}$ the null matrix/null vector of the appropriate order.

(a) Calculate $\mathbf{U}^2$ and $(r\mathbf{L} + \mathbf{D} + u\mathbf{U})\mathbf{L}$.

(b) Put $s = t = 1$. In this part, you shall solve for the inverse of $(\mathbf{D} + u\mathbf{U})$ or show that it does not exist, for every $u > 0$. That is, you shall solve the equation system $\mathbf{AX} = \mathbf{I}$, where $\mathbf{A} = \mathbf{D} + u\mathbf{U}$. Note that the unknown $\mathbf{X}$ is 5×5.

- For full score:
  Write down the augmented coefficient matrix $(\mathbf{A} \vdots \mathbf{I})$ and perform Gaussian elimination on this until you have found the solution or shown that no solution exists.
- For up to 2/3 score (usually corresponding to a near-middle-of-the-road "C"):
  Solve the three equation systems $\mathbf{Ax} = \big(1,\ 0,\ 0,\ 0,\ 0\big)'$, $\mathbf{Ay} = \big(0,\ 1,\ 0,\ 0,\ 0\big)'$ and $\mathbf{Az} = \big(0,\ 0,\ 1,\ 0,\ 0\big)'$ and explain then where/how $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ enters the inverse and how you would solve the rest.

(c) Let $\mathbf{A}$ and $\mathbf{B}$ be matrices such that $\mathbf{AB} = \mathbf{BA}$ – that is, we say that the matrices $\mathbf{A}$ and $\mathbf{B}$ *commute.* Decide which of the following statements is the true one:

(I) $\mathbf{A}'$ and $\mathbf{B}'$ always commute.

(II) $\mathbf{A}'$ and $\mathbf{B}'$ commute if both are square; otherwise, there exist counterexamples.

(III) Even if both are symmetric, there are cases where they do not commute, provided the order is large enough (that is, that n is large enough if they are $n \times n$; they obviously commute when $n = 1$!)

(d) Let $\mathbf{1}$ denote the n-vector of ones. Suppose that the $n \times n$ matrix $\mathbf{M}$ is such that $\mathbf{M1} = \lambda\mathbf{1}$ for some number λ. How many solutions will then the equation system $(\mathbf{M} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ have?
Hint: Write out $(\mathbf{M} - \lambda\mathbf{I})\mathbf{x}$; when is that difference $= \mathbf{0}$?