The Leibniz rule: ECON3120/4120 curriculum

About this document: The Leibniz rule for differentiating integrals is now curriculum. This note clarifies what to expect (this semester!).

What you need to *know*: Let $a = a(x)$, $b = b(x)$ and $f = f(x,t)$ be given functions*. If $\quad F(x) = \displaystyle\int_{a(x)}^{b(x)} f(x,t)\,dt \quad$ (note: integration wrt. t), then

$$F'(x) = f(x, b(x)) \cdot b'(x) - f(x, a(x)) \cdot a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt \qquad (*)$$

In this Wikipedia permalink† the "curriculum" part stops before the "Contents" table.

What you need to be able to *do*: Calculate derivatives when you are asked and when you need it, just like any other differentiation rule.

Why it works? An application of the chain rule: A function $H(x,y,z)$ has differential $dH = H'_x\,dx + H'_y\,dy + H'_z\,dz$, and if $y = a(x)$ and $z = b(x)$, then $dy = a'(x)\,dx$ and $dz = b'(x)\,dx$, so that $F(x) = H(x, a(x), b(x))$ has differential $dF = \Big(H'_x + H'_y a' + H'_z b'\Big)\,dx$. Let $H(x,y,z) = \int_y^z f(x,t)\,dt$ and calculate partial derivatives.

- For H'_z, note that x is treated as constant when we integrate with respect to t. If $G'(t) = g(t)$, then $\int_y^z g(t) = G(z) - G(y)$, and the derivative wrt. z is $G'(z) = g(z)$. If we now introduce a constant in the notation and write $f(x,t)$ for this "$g(t)$ with x as parameter", then it is the t variable that should be put equal to z. So

$$H'_z(x,y,z) = f(x,z)$$

- Analogously, $H'_y(x,y,z) = -f(x,y)$ because y is the lower limit (recall $\int_y^z = -\int_z^y$).
- From the definition of partial derivatives, $H'_x(x,y,z)$ equals

$$\frac{\partial}{\partial x}\int_y^z f(x,t)\,dt = \lim_{\epsilon\to 0} \frac{\int_y^z f(x+\epsilon,t)\,dt - \int_y^z f(x,t)\,dt}{\epsilon} = \lim_{\epsilon\to 0} \int_y^z \frac{f(x+\epsilon,t) - f(x,t)}{\epsilon}\,dt$$

  and by moving the "lim" inside the integral sign‡, the integrand becomes $\frac{\partial}{\partial x} f(x,t)$.

Inserting in $\; F' = H'_x + H'_y\,a' + H'_z\,b' \;$ yields the three terms on the right-hand side of $(*)$.

*in Mathematics 2, they will be assumed "sufficiently nice", and no precise conditions will be given

† `https://en.m.wikipedia.org/w/index.php?title=Leibniz_integral_rule&oldid=804035562` for mobile-view version or for copy/paste. Note the permalink, immune to Wikipedia edits.

‡ *yes,* you are allowed to do that in Mathematics 2, as all functions are nice enough; *no,* it is not "obvious"

Examples/exercises: This used to be Mathematics 3 curriculum, so problems can be found in the Mathematics 3 compendium,[§] The following are just examples:

(1) Let $F(x) = \int_{x^2}^{x} te^{te^x}\,dt$. Show that $F'(0) = 0$, with and without using Leibniz's rule.
Answer: By the Leibniz rule, $b(0)e^{b(0)e^0}b'(0) - a(0)e^{a(0)e^0}a'(0) + \int_{a(0)}^{b(0)} \left[te^{te^x} \cdot te^x\right]_{x=0} dt$. But $b(0) = a(0) = 0$, so all the terms are zero (the integral because it is $\int_0^0$). Solving without: exercise. Take note that it is a bit more work, maybe?

(2) Let $F(x)$ as in Example 1. Find $F''(0)$.
Answer: The formula yields $F'(x) = b(x)e^{b(x)e^x}b'(x) - a(x)e^{a(x)e^x}a'(x) + \int_{a(x)}^{b(x)}[te^{te^x} \cdot te^x]dt$, with $b(x) = x$ and $a(x) = x^2$; so, $F'(x) = xe^{xe^x} - 2x^3e^{x^2e^x} + \int_{x^2}^{x}[te^{te^x} \cdot te^x]\,dt$. The derivative at zero of xe^{xe^x} is $\lim_{x\to 0} \frac{xe^{xe^x}}{x} = 1$ (exercise (i): what did I just do?) and the derivative zero of $2x^3e^{x^2e^x}$ is $\lim_{x\to 0} 2x^2e^{x^2e^x} = 0$ (exercise (ii): as (i) – what did I just do?) To differentiate the integral term, use the Leibniz rule again. Exercise (iii): show that you get zero from that term, so the answer is 1.[¶]

(3) This you "need" the Leibniz rule for: Find $\frac{d}{dx}\int_1^e t^{-1}e^{(1+x^2)t}\,dt$.
Answer: We get $\int_1^e t^{-1}\frac{\partial}{\partial x}e^{(1+x^2)t}\,dt = \int_1^e 2xe^{(1+x^2)t}\,dt$. Notice that now the bothersome t^{-1} is gone! The rest is routine: $\frac{2x}{1+x^2}\Big|_{t=1}^{t=e} e^{(1+x^2)t} = \frac{2x}{1+x^2}\left[e^{(1+x^2)e} - e^{1+x^2}\right]$

(4) More generally: Given continuously differentiable functions a, b and w, which do never hit zero. Explain how to calculate $\frac{d}{dx}\int_{a(x)}^{b(x)} t^{-1}e^{tw(x)}\,dt$.
Answer: We get $e^{b(x)w(x)}b'(x)/b(x) - e^{a(x)w(x)}a'(x)/a(x) + w'(x)\int_{a(x)}^{b(x)} e^{tw(x)}\,dt$. The latter integral is solved by the substitution $u = tw$.

(5) Let $p(t) = \dot{p}(t) + \int_0^{\sqrt{p(t)}} e^{q^2-p(t)}\,dq$. Deduce (but do *not* try to solve!) a second-order differential equation for p *without* evaluating the integral.
Answer: Beware that now the variable to be integrated is called q! We have $\dot{p} = \ddot{p} + e^{p-p}\cdot\frac{\dot{p}}{2\sqrt{p}} + \int_0^{\sqrt{p}} e^{q^2-p}\cdot(-\dot{p})\,dq$. The last term is $-\dot{p}\cdot(p-\dot{p})$. Gathering terms, we get $\left(1 + p - \frac{1}{2\sqrt{p}}\right)\dot{p} - (\dot{p})^2 = \ddot{p}$.
Alternatively: rewrite as $\dot{p} + e^{-p}\int_0^{\sqrt{p}} e^{q^2}dq$. Then $\dot{p} = \ddot{p} - \dot{p}e^{-p}\int_0^{\sqrt{p}} e^{q^2}dq + e^{-p}\frac{d}{dt}\int_0^{\sqrt{p}} e^{q^2}dq$. The second term is $-\dot{p}(p-\dot{p})$ again. For the third, use Leibniz's rule: $e^{-p}e^{p}\cdot\frac{d}{dt}\sqrt{p(t)}$.

[§] `http://www.uio.no/studier/emner/sv/oekonomi/ECON4140/oldexams/M3Compendium.pdf` problems 4-01/02/10/11 (and, possibly with some hints, -03 and -04, which involve "unknown" functions).

[¶] An alternative, and somewhat more sophisticated approach on the integral, using the intermediate value theorem: for fixed x, the integral is $T^2e^{x+Te^x}\cdot[x - x^2]$ for some T between x^2 and x (make the "area under graph" argument, where the interval has width $x^2 - x$). Divide by x (as we seek the limit of $F'(x)/x$ as per exercises (i) and (ii)) to get $T^2e^{x+Te^x}\cdot[1-x]$. T is not known when $x > 0$, but is squeezed between x^2 and x, and must $\to 0$ as $x \to 0$ (like an argument in the last part of Term Paper Problem 2!). So we get 0 from the integral and again, we are left with only the contribution $\lim_{x\to 0} b(x)e^{b(x)e^x}b'(x) = 1$.