LA lecture 2: Matrices: what for?

What? a rectangular array of m rows and n columns of numbers.

What is it good for? (Putting numbers in boxes, huh? Apart from compact notation as bookkeeping tool?)

- We shall write the linear equation system

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned} \qquad \text{as } \mathbf{A}\mathbf{x} = \mathbf{b}$$

- Yesterday: manipulation tools for vectors (budgets, ...).
  Now: manipulation tools for matrices, and then: use them to solve and/or characterize the solution/solvability of $\mathbf{A}\mathbf{x} = \mathbf{b}$.

(Other courses? If a random vector $\mathbf{X}$ has covariance matrix $\mathbf{W}$, then the random variable $Y = \mathbf{c} \cdot \mathbf{X}$ (with $\mathbf{c}$ nonrandom) has variance $\mathbf{c} \cdot (\mathbf{W}\mathbf{c})$ ...)

LA lecture 2: Matrices: definition

Definition: a matrix of *order* $m \times n$ (read: "m by n") is a rectangular array of m rows and n columns of numbers.

- **Example:** This example matrix is 2×3:

$$\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$$

- **Elements:** The numbers, indexed by (rownumber, column number), indexing counted from top–left.
- **Notation:** We write a_{ij} for the elements of $\mathbf{A}$. ("$a_{i,j}$" if needed). Example: for $\mathbf{H}$ above, $h_{21} = 2e$.
- **Specification by elements:** We can specify a matrix by specifying the elements individually.
  Examples: write down the 3×2 matrices $\mathbf{U}$ and $\mathbf{V}$ defined by $u_{ij} = i - j$ and $v_{ij} = (-1)^{i+j}$.

LA lecture 2: Matrices: equality, transpose

from previous page:

$$\mathbf{U} = \begin{pmatrix} 1-1 & 1-2 \\ 2-1 & 2-2 \\ 3-1 & 3-2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Equality: element-wise. We have $\mathbf{A} = \mathbf{B}$ iff $a_{ij} = b_{ij}$, all i, j (orders must be the same).

Definition: the *transpose* $\mathbf{A}'$ of the $m \times n$ matrix $\mathbf{A} = (a_{ij})$ is $n \times m$ with elements $b_{ij} = a_{ji}$. We call $\mathbf{A}$ *symmetric* if $\mathbf{A}' = \mathbf{A}$.

[The prime symbol is not a derivative. No confusion as long as we keep linear algebra and analysis separated!]

- **Example:** if $\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$, then $\mathbf{H}' = \begin{pmatrix} 2018 & 2e \\ 9 & -1.4 \\ 25 & 5 \end{pmatrix}$
- **Exercise:** Explain why $(\mathbf{A}')' = \mathbf{A}$.

LA lecture 2: Vectors as matrices. Rows & columns.

Vectors recast: A row vector is a $1 \times n$ matrix. A column vector is an $m \times 1$ matrix.

- The transpose of a row vector is a column vector.
  The transpose of a column vector is a row vector.
- **Vectors default to columns** from now on.
  To specify a row vector, I will use a prime.
  - **Example:** The first row of the example matrix $\mathbf{H}$ is $\mathbf{r}_1' = (2018 \;\; 9 \;\; 25)$. Here, $\mathbf{r}_1$ is a column, namely $\mathbf{r}_1 = \begin{pmatrix} 2018 \\ 9 \\ 25 \end{pmatrix}$.
  - To save space, you can specify a column $\mathbf{x}$ as e.g. $\mathbf{x} = (1 \; 2 \; 3)'$. (Or, comma-separated.)

Noticed? We can specify a matrix by its rows or its columns.

- **Example:** $\mathbf{H}$ is given by its rows $\mathbf{r}_1' = (2018 \;\; 9 \;\; 25)$ and $\mathbf{r}_2 = (2e \;\; -1.4 \;\; 5)$. (*Enumeration matters!!*) Alternatively, by its columns $\mathbf{c}_1 = \begin{pmatrix} 2018 \\ 2e \end{pmatrix}$, $\mathbf{c}_2 = \begin{pmatrix} 9 \\ -1.4 \end{pmatrix}$ and $\mathbf{c}_3 = \begin{pmatrix} 25 \\ 5 \end{pmatrix}$.

LA lecture 2: square matrices will turn out quite significant

Definitions:

- A matrix $\mathbf{S}$ is *square* if it is $n \times n$. The elements s_{ii} (i.e., s_{ij} with $i = j$) are called the *main diagonal* elements.
- A (necessarily square) matrix $\mathbf{S}$ is called *symmetric* if $\mathbf{S}' = \mathbf{S}$.
- A (necessarily symmetric) matrix $\mathbf{D}$ is *diagonal* if $d_{ij} = 0$ whenever $i \neq j$ (the "off-diagonal" elements are zero)
- The *identity matrix* $\mathbf{I}_n$ of order $n \times n$ is diagonal with elements $= 1$ on the main diagonal.

$$\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}; \qquad \begin{pmatrix} 14 & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(asterisks indicate the main diagonal; a diagonal matrix; $\mathbf{I}_3$.)

We often say "order n identity" rather than "$n \times n$", and write $\mathbf{I}$ without subscript if order is understood.

LA lecture 2: Matrices: scaling and addition.

First:

Definition: The order $m \times n$ **zero (/null) matrix** $\mathbf{0}_{mn}$ (denoted simply $\mathbf{0}$ if order is understood), has all elements equal to zero.

Scaling is defined element-wise: $\mathbf{C} = \alpha\mathbf{B}$ has elements $c_{ij} = \alpha b_{ij}$.

Addition is defined element-wise, provided the matrices have the same order: $\mathbf{C} = \mathbf{A} + \mathbf{B}$ has elements $c_{ij} = a_{ij} + b_{ij}$.

Scaling and addition "follow nice rules", like for vectors. Assuming all matrices are of same order $m \times n$, then

$(\alpha + \beta)(\mathbf{A} + \lambda\mathbf{B} + \mathbf{0}) = (\alpha\lambda + \beta\lambda)\mathbf{B} + \alpha\mathbf{A} + \beta\mathbf{A}$

$\alpha\mathbf{0} = \mathbf{0}$ and $0\mathbf{A} = \mathbf{0}$ and $1\mathbf{A} = \mathbf{A}$

Subtraction: $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$.

Downscaling: $\frac{1}{\alpha}\mathbf{A}$ is OK for $\alpha \neq 0$.

and for transposition: $(\alpha\mathbf{A})' = \alpha\mathbf{A}'$ and $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$.

The matrix product $\mathbf{AB}$ is defined only iff

> *the number of columns of (the left)* $\mathbf{A}$ *equals*
> *the number of rows of (the right)* $\mathbf{B}$.

That is: $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$. (Note where the "n" occurs!)

Definition: The product $\mathbf{C} = \mathbf{AB}$ of an $m \times n$ matrix $\mathbf{A}$ and $n \times p$ matrix $\mathbf{B}$, is $m \times p$ with

$$c_{ij} = \mathbf{r}_i \cdot \mathbf{k}_j, \qquad \text{where}$$

$\mathbf{r}_i'$ is the ith row of $\mathbf{A}$, and

$\mathbf{k}_j$ is the kth column of $\mathbf{B}$.

Examples / not? Let $\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$ and $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Which are *well-defined* of $\mathbf{I}_2\mathbf{I}_2$, $\cancel{\mathbf{H}\mathbf{I}_2}$, $\mathbf{I}_2\mathbf{H}$, $\cancel{\mathbf{H}\mathbf{H}}$, $\mathbf{H}\mathbf{H}'$, $\mathbf{H}'\mathbf{H}$?

$\underbrace{2\times 2 \quad 2\times 2}_{\text{OK}}$ $\quad$ $2\times 3 \quad 2\times 2$ $\quad$ $2\times 2 \quad 2\times 3$ $\quad$ $2\times 3 \quad 2\times 3$ $\quad$ $2\times 3 \quad 3\times 2$ $\quad$ $3\times 2 \quad 2\times 3$

LA preview 2: matrix multiplication II

"bottom – leftmost" is not standard terminology, but : (this) last row of left matrix • first column of right matrix

Cont'd: Let $\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$ and $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Of each of those which are well-defined among $\mathbf{I}_2\mathbf{I}_2$, $\cancel{\mathbf{H}\mathbf{I}_2}$, $\mathbf{I}_2\mathbf{H}$, $\cancel{\mathbf{H}\mathbf{H}}$, $\mathbf{H}\mathbf{H}'$, $\mathbf{H}'\mathbf{H}$: calculate the "bottom–leftmost" element.

$\vec{I_2}\vec{I_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

element $= (0\ 1)\cdot(1\ 0) = \underline{\underline{0}}$

$\vec{I_2}\vec{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$

element $= (0\ 1)\cdot(2018\ 2e) = \underline{\underline{2e}}$

$\vec{H}\vec{H}' = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}\begin{pmatrix} 2018 & 2e \\ 9 & -1.4 \\ 25 & 5 \end{pmatrix}$

so: element $= 4036e - 12{,}6 + 125 = \ldots$

$\vec{H}'\vec{H} = \begin{pmatrix} 2018 & 2e \\ 9 & -1.4 \\ 25 & 5 \end{pmatrix}\begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$

element $= 50450 + 10e$

Example (small): Could $\mathbf{AB}$ be a 1×1 matrix? Hint: dot product? ⊗

Yes. $1 \times n$ $n \times 1$.

Example ("big"?): Calculate $\mathbf{AB}$ where $\mathbf{A} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2 & \ldots & 2 \end{pmatrix}$ has 2018 columns and $\mathbf{B}$ is 2018×3 with all elements b_{ij} equal to 1.

⟸ next handwritten

These calculations need a bit more space :-o

⊗ Let $\vec{x}$ and $\vec{y}$ be column vectors (in $\mathbb{R}^n$, same n). both $n\times 1$

Then $\underset{1\times n}{\vec{x}'}\ \underset{n\times 1}{\vec{y}} = \vec{x}\cdot\vec{y}$.

sometimes you only see matrix products, no "dots"!

$\vec{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \end{pmatrix}$ is 2×2018,

$\vec{B}$ is 2018×3, all $b_{ij} = 1$.

$\vec{A}\vec{B} = ?$

Note: $\vec{B} = (\vec{k}_1 \,\vdots\, \vec{k}_2 \,\vdots\, \vec{k}_3)$, $\boxed{\vec{A}\vec{B} = (\vec{A}\vec{k}_1 \,\vdots\, \vec{A}\vec{k}_2 \,\vdots\, \vec{A}\vec{k}_3)}$

~~row~~ column j = $\vec{A}\vec{k}_j$

This $\vec{B}$: $\vec{k}_1 = \vec{k}_2 = \vec{k}_3 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Also: $\vec{A} = \begin{pmatrix} \vec{r}_1{}' \\ \vec{r}_2{}' \end{pmatrix}$; this $\vec{A}$: $\begin{pmatrix} \vec{r}\,' \\ 2\vec{r}\,' \end{pmatrix}$

By $\vec{r}_2{}' = 2\vec{r}_1{}'$ and $\vec{k}_1 = \vec{k}_2 = \vec{k}_3$, $\vec{A}\vec{B}$ will be

$\begin{pmatrix} t & t & t \\ 2t & 2t & 2t \end{pmatrix}$ where $t = \cancel{\vec{k}_1 \cdot \vec{A}} = \cancel{1}$

this $\vec{A}$ this $\vec{B}$

$\vec{1} \cdot \vec{k}_1 = \vec{1}{}'\,\vec{1} = 2018.$

row · col

$$\vec{A}\vec{B} = \underline{\underline{2018 \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}}}$$

LA lecture 2: matrix multiplication “howto”

What is the third row of the following matrix product? What is the fourth column? Then, start calculating from top–left:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 & 0 \\ 4 & 2 & 0 & 0 \\ 4 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} ? & ? & ? & 0 \\ ? & ? & ? & 0 \\ 0 & 0 & 0 & 0 \\ ? & ? & ? & 0 \end{pmatrix}$$

How to calculate? For the first column of the product, you need the left matrix and the first column of the right:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 4 & \\ 4 & \text{[redacted]} \\ 4 & \end{pmatrix} = \begin{pmatrix} 4+4+4 & \\ 0+4+0 & \text{[redacted]} \\ 0+0+0 & \\ 16-20+24 & \end{pmatrix}$$

Then on to the second column. Though you know a few “0” elements.

Second column: $\begin{pmatrix}1&1&1\\0&1&0\\0&0&0\\4&-5&6\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1+2+3\\0+2+0\\0+0+0\\4-10+18\end{pmatrix} = \begin{pmatrix}6\\2\\0\\12\end{pmatrix}$

Third column: [you get it?] (Fourth $= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$, known.)

[Said in lecture, not written:]

Q: What would have happened if we put another row $\vec{r}\,'$ atop the left matrix?

A: Product would get another row $\vec{r}\,' \begin{pmatrix}4&2&1&0\\4&2&0&0\\4&3&1&0\end{pmatrix}$ on top.

Q: What about $\begin{pmatrix}1&1&1\\0&1&0\\0&0&0\\4&-5&6\end{pmatrix}\begin{pmatrix}4&1&2&5\\4&2&0&5\\4&3&1&5\end{pmatrix}$?

A: First three columns unchanged. Fourth: $\begin{pmatrix}5\\5\\5\end{pmatrix} = \frac{5}{4}\begin{pmatrix}4\\4\\4\end{pmatrix}$ ← first of right

so the fourth column becomes $\frac{5}{4}\begin{pmatrix}12\\4\\0\\20\end{pmatrix} = \begin{pmatrix}15\\5\\0\\25\end{pmatrix}$.

Rules: Let α and β be numbers. Suppose $\mathbf{A}$ being $m \times n$, and suppose for each formula that $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ have orders such that sums and products *are well-defined.* Then:

$\mathbf{0}_{k,m}\mathbf{A} = \mathbf{0}_{k,n}$ and $\mathbf{A}\mathbf{0}_{n,p} = \mathbf{0}_{m,p}$. (Note: orders of "$\mathbf{0}$"!)

$\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$ (Note: orders of "$\mathbf{I}$"!)

$(\alpha\mathbf{A})(\beta\mathbf{B}) = (\alpha\beta)\mathbf{A}\mathbf{B}$, we drop the parentheses: $\alpha\beta\ \mathbf{A}\mathbf{B}$.

$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{A}\mathbf{B}+\mathbf{A}\mathbf{C}$ and $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}+\mathbf{B}\mathbf{C}$

- Note: $\mathbf{A}\mathbf{B}+\beta\mathbf{B} = (\mathbf{A}+\beta\mathbf{I}_n)\mathbf{B}$ ($\mathbf{A}$ necessarily $n \times n$. Take care *not* to write the ill-defined "$\mathbf{A}$ plus β" times $\mathbf{B}$.)

$\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}$, we drop the parentheses: $\mathbf{A}\mathbf{B}\mathbf{C}$.

$(\mathbf{A}\mathbf{B})' = \mathbf{B}'\mathbf{A}'$, so also $(\mathbf{A}\mathbf{B}\mathbf{C})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$

Multiplication of squares: $\mathbf{A}\mathbf{A}$ exists iff $\mathbf{A}$ is square. For square matrices, we write $\mathbf{A}^k$ for the k-fold product $\mathbf{A}\mathbf{A}\cdots\mathbf{A}$ $(k \in \mathbb{N})$.

Small exercise: Explain why $\mathbf{A}\mathbf{A}'$ always exists and is symmetric.

LA lecture 2: matrix multiplication: INVALID operations

Take care not to apply bogus rules:

- Matrix multiplication is *not* performed element-wise, not even when $\mathbf{A}$, $\mathbf{B}$ both $n \times n$. (Exercise: what if both are diagonal?)
- Except "by coincidence", $\mathbf{AB} \neq \mathbf{BA}$.
  - Even when both products are well-defined and of the same order – i.e., both $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ – the products are usually unequal. (Calculate: $\mathbf{A} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\mathbf{B} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ ... ?)
  - Exercise: for numbers we have $\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$ and formulae for squares of sums/differences – are they valid if α and β are replaced by $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$?
- Do not divide by matrices! Leave $\mathbf{AC} = \mathbf{DC}$ as-is ... for now.
  - Later: criteria for when that is indeed $\iff \mathbf{A} = \mathbf{D}$. But even then, you cannot slash $\mathbf{C}$ off $\mathbf{CA} = \mathbf{BC}$ nor from $\mathbf{ACA} = \mathbf{BCB}$.
  - (But 1×1s that are (non-zero!) numbers? ... ?)
- It is possible that $\mathbf{A}^2 = \mathbf{0}$ even when all $a_{ij} \neq 0$. Example: $\mathbf{A} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. (But $\mathbf{A}'\mathbf{A} \neq \mathbf{0}$ for $\mathbf{A} \neq \mathbf{0}$, cf. dot product.)

LA lecture 2: linear transformations and eq. systems

Terminology: multiplication "does not commute"; Fix $\mathbf{C}$. To get $\mathbf{LCR}$, we "left-multiply by $\mathbf{L}$" and "right-multiply by $\mathbf{R}$".
(Alternative phrases: pre-multiply/post-multiply.)

Matrix multiplication can be thought of as linear transformation, and the *only* linear transformatios ("functions") from $\mathbb{R}^n$ to $\mathbb{R}^m$, are by some matrix multiplication taking $\mathbf{x}$ in and returning $\mathbf{Ax}$.

- The Math2-relevant consequence: The only possible linear equations for n unknowns $\mathbf{x}$, are of the form $\mathbf{Ax} = \mathbf{b}$.
- Next: an algorithm to solve. Before that: give me *the truth, the whole truth, and nothing but the truth* about the solution of the single-variable linear equation (for x) $$ax = b$$

  (Hint: Saying "$x = a^{-1}b$" is not good enough.)